Protected: POW 2012-1 ArcTan

Compute $\tan^{-1}(1) - tan^{-1}(1/3) + tan^{-1}(1/5) - tan^{-1}(1/7) + \cdots$.

Solution. Let's remind the following well-known theorem without a proof.

Thm. For all $z \in \mathbb{C}$,

$$\sin z = z \prod_{k=1}^{\infty}(1 - \frac{z^2}{\pi^2 k^2}).$$

As a corollary,

Cor. For all $z, a \in \mathbb{C}$ with $a \notin \pi\mathbb{Z}$,

$$\frac{\sin(z+a)}{\sin a} = \frac{z+a}{a} \prod_{k=1}^{\infty}(1 - \frac{z}{k\pi - a})(1 + \frac{z}{k\pi + a}).$$

Now, I want to derive the final result from the above corollary:

Formula. For all $x, a \in \mathbb{R}$ with $a \notin \pi\mathbb{Z}$,

$$\tan^{-1}(\tanh x \cot a) = \tan^{-1}\frac{x}{a} + \sum_{k=1}^{\infty}(\tan^{-1}(\frac{x}{k\pi + a}) - \tan^{-1}(\frac{x}{k\pi - a})).$$

proof. By the corollary, $\Im(\ln \frac{\sin(a+ix)}{\sin a})$

$$= \Im(\ln \left((1 + \frac{ix}{a}) \prod_{k=1}^{\infty}(1 - \frac{ix}{k\pi - a})(1 + \frac{ix}{k\pi + a}) \right))$$

$$= \tan^{-1}\frac{x}{a} + \sum_{k=1}^{\infty}(\tan^{-1}(\frac{x}{k\pi + a}) - \tan^{-1}(\frac{x}{k\pi - a})) + m\pi$$

for some $m \in \mathbb{Z}$. On the other hand,

$$\Im(\ln \frac{\sin(a+ix)}{\sin a}) = \Im(\cosh x + i \cot a \sinh x) = \tan^{-1}(\tanh x \cot a) + n\pi$$ for some $n \in \mathbb{Z}$.

Thus, it is enough to show that $m = n$ for all x. Note that this is true for $x = 0$. It is not so hard to show that RHS of the formula is continuous using the Weierstrass' M-test. As LHS of the formula is also continuous, m must be same with n for all x. Therefore, the formula is verified.

Finally, put $x = a = \frac{\pi}{4}$ to our formula to obtain

$$\tan^{-1}(1) - tan^{-1}(1/3) + tan^{-1}(1/5) - tan^{-1}(1/7) + \cdots = \tan^{-1} \tanh \frac{\pi}{4}.$$