Problem of the Week 2012-2

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Problem: Let n be a positive integer and let S_n be the set of all permutations on $\{1, 2, \ldots, n\}$. Assume $x_1 + x_2 + \cdots + x_n = 0$ and $\sum_{i \in A} x_i \neq 0$ for all nonempty proper subsets A of $\{1, 2, \ldots, n\}$. Find all possible values of

$$\sum_{\pi \in S_n} \frac{1}{x_{\pi(1)}} \frac{1}{x_{\pi(1)} + x_{\pi(2)}} \cdots \frac{1}{x_{\pi(1)} + \cdots + x_{\pi(n-1)}} .$$

Solution:

We see that

$$\sum_{\pi \in S_n} \frac{1}{x_{\pi(1)}} \frac{1}{x_{\pi(1)} + x_{\pi(2)}} \cdots \frac{1}{x_{\pi(1)} + \cdots + x_{\pi(n-1)}}$$
$$= \sum_{\pi \in S_n} \prod_{m=1}^{n-1} \frac{1}{\sum_{i=1}^{m} x_{\pi(i)}} .$$

The second expression has a consistent definition for $n = 1$, where it is reduced to 1. On the other hand, for $n \geq 2$, claim 1 (to be stated and proved below) along with the condition that $x_1 + x_2 + \cdots + x_n = 0$ lead to

$$\sum_{\pi \in S_n} \prod_{m=1}^{n-1} \frac{1}{\sum_{i=1}^{m} x_{\pi(i)}} = 0 \quad (n \geq 2) .$$

Therefore, the only possible values of the expression given in this problem are 0 and 1.

Claim 1. Let n be an integer such that $n \geq 2$ and $\sum_{i \in A} x_i \neq 0$ for all nonempty proper subsets A of $\{1, 2, \ldots, n\}$. Then,

$$\sum_{\pi \in S_n} \prod_{m=1}^{n-1} \frac{1}{\sum_{i=1}^{m} x_{\pi(i)}} = \frac{\sum_{i=1}^{n} x_i}{\prod_{i=1}^{n} x_i} .$$

Proof of claim 1.

We prove this using mathematical induction on n.

(i) Claim 1 holds for $n = 2$ because

$$\begin{aligned}(\text{L.H.S.}) &= \frac{1}{x_1} + \frac{1}{x_2}\,,\\ (\text{R.H.S.}) &= \frac{x_1 + x_2}{x_1 x_2} = \frac{1}{x_1} + \frac{1}{x_2}\,.\end{aligned}$$

(ii) Assume that the claim holds for $n = k$. Now consider the case where $n = k+1$. Among the terms in the summation over $\pi \in S_{k+1}$, group together those with the same value of $\pi(k+1)$. For a fixed $\pi(k+1) = p$, we are effectively considering all permutations of $\{1, \ldots, p-1, p+1, \ldots, k+1\}$, so let us define

$$y_i^{(p)} \equiv \begin{cases} x_i & (i < p) \\ x_{i+1} & (i \geq p) \end{cases}\,.$$

Then,

$$\begin{aligned}\sum_{\pi \in S_{k+1}} \prod_{m=1}^{(k+1)-1} \frac{1}{\sum_{i=1}^{m} x_{\pi(i)}} &= \sum_{p=1}^{k+1} \sum_{\pi' \in S_k} \prod_{m=1}^{k} \frac{1}{\sum_{i=1}^{m} y_{\pi'(i)}^{(p)}}\\ &= \sum_{p=1}^{k+1} \frac{1}{\sum_{i=1}^{k} y_i^{(p)}} \sum_{\pi' \in S_k} \prod_{m=1}^{k-1} \frac{1}{\sum_{i=1}^{m} y_{\pi'(i)}^{(p)}}\\ &= \sum_{p=1}^{k+1} \frac{1}{\prod_{i=1}^{k} y_i^{(p)}} = \sum_{p=1}^{k+1} \prod_{\substack{1 \leq i \leq k+1 \\ i \neq p}} \frac{1}{x_i}\\ &= \frac{\sum_{i=1}^{k+1} x_i}{\prod_{i=1}^{k+1} x_i}\,,\end{aligned}$$

To obtain the third equality, we have used the assumption that the claim is valid for $n = k$. Now we have shown that claim 1 holds for $n = k + 1$.

By (i) and (ii) above, claim 1 is true for any integer n satisfying $n \geq 2$. Q. E. D.

An alternative method to prove claim 1.

Consider the following equation:

$$
\begin{aligned}
\frac{1}{\prod_{i=1}^n x_i} &= \int_0^\infty dt_1 \int_0^\infty dt_2 \cdots \int_0^\infty dt_n \exp\left(-\sum_{m=1}^n x_m t_m\right) \\
&= \sum_{\pi\in S_n} \int_0^\infty dt_{\pi(1)} \int_0^{t_{\pi(1)}} dt_{\pi(2)} \cdots \int_0^{t_{\pi(n-1)}} dt_{\pi(n)} \exp\left(-\sum_{m=1}^n x_m t_m\right) \\
&= \sum_{\pi\in S_n} \int_0^\infty dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \exp\left(-\sum_{m=1}^n x_{\pi(m)} t_m\right) \\
&= \sum_{\pi\in S_n} \int_0^\infty d\tilde{t_1} \int_0^\infty d\tilde{t_2} \cdots \int_0^\infty d\tilde{t_n} \left|\frac{\partial(t_1,\ldots,t_n)}{\partial(\tilde{t}_1,\ldots,\tilde{t}_n)}\right| \exp\left[-\sum_{m=1}^n \left(\sum_{i=1}^m x_{\pi(i)}\right)\tilde{t}_m\right] \\
&= \sum_{\pi\in S_n} \prod_{m=1}^n \frac{1}{\sum_{i=1}^m x_{\pi(i)}} \\
&= \frac{1}{\sum_{i=1}^n x_i} \sum_{\pi\in S_n} \prod_{m=1}^{n-1} \frac{1}{\sum_{i=1}^m x_{\pi(i)}} .
\end{aligned}
$$

In the second line, we have divided the region of integration into $n!$ different parts, where each of them corresponds to the case where $t_{\pi(1)} > t_{\pi(2)} > \ldots > t_{\pi(n)}$ ($\pi \in S_n$). Next, we have relabeled the integration variables by $t_{\pi(i)} \to t_i$. Then, we have made a change of integration variables such that $\tilde{t_n} = t_n$ and $\tilde{t_i} = t_i - t_{i+1}$ ($i < n$), which gives $t_i = \sum_{j=i}^n \tilde{t}_j$. The Jacobian matrix is an upper triangular matrix whose nonzero elements are all equal to 1, and hence its determinant is simply 1.

Our analysis up to now is sufficient as a motivation for claim 1, but there is a problem if we actually try to prove it following the same lines. The reason is that the integrations are well-defined only when $\mathrm{Re}(x_i) > 0$ for all i. To avoid this pathology, we first make a replacement $x_i \to z + x_i$ to obtain

$$
\frac{1}{\prod_{i=1}^n (z+x_i)} = \frac{1}{nz + \sum_{i=1}^n x_i} \sum_{\pi\in S_n} \prod_{m=1}^{n-1} \frac{1}{mz + \sum_{i=1}^m x_{\pi(i)}} .
$$

Again, $\mathrm{Re}(z+x_i) > 0$ should be satisfied for all i, for the intermediate steps to make sense. However, from both sides of the above relation, we can analytically continue z to the whole complex plane aside from a finite number of poles. Now we can multiply both sides by $nz + \sum_{i=1}^n x_i$ and take the limit where $z \to 0$, to prove claim 1.