POW 2012-2 Sum with a permutation

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Let $S = \sum_{\pi \in S_n} \frac{1}{x_{\pi(1)}} \frac{1}{x_{\pi(1)} + x_{\pi(2)}} \cdots \frac{1}{x_{\pi(1)} + x_{\pi(2)} + \dots + x_{\pi(n-1)}}$ (①)

Since permutations are bijective functions from {1,2,...,n} to {1,2,...,n} so there is an inverse of each permutation in S_n.

We can apply a permutation $\pi_1 = \begin{pmatrix} 1\ 2\ 3 \dots n \\ 2\ 1\ 3 \dots n \end{pmatrix}$ to all elements of S_n, and get S_n as a new expression

$$S_n = \{\pi\pi_1 | \pi \in S_n\}$$

because $\forall \pi \in S_n$, $\pi' = \pi\pi_1^{-1} \in S_n$ satisfies $\pi = \pi'\pi_1$

Then we deduce

$$S = \sum_{\pi \in S_n} \frac{1}{x_{\pi(2)}} \frac{1}{x_{\pi(2)} + x_{\pi(1)}} \cdots \frac{1}{x_{\pi(1)} + x_{\pi(2)} + \dots + x_{\pi(n-1)}} \quad (②)$$

By adding equation ① and ②,

$2S = \sum_{\pi \in S_n} \left(\frac{1}{x_{\pi(1)}} + \frac{1}{x_{\pi(2)}} \right) \frac{1}{x_{\pi(1)} + x_{\pi(2)}} \cdots \frac{1}{x_{\pi(1)} + x_{\pi(2)} + \dots + x_{\pi(n-1)}}$

$= \sum_{\pi \in S_n} \frac{1}{x_{\pi(1)} x_{\pi(2)}} \frac{1}{x_{\pi(1)} + x_{\pi(2)} + x_{\pi(3)}} \cdots \frac{1}{x_{\pi(1)} + x_{\pi(2)} + \dots + x_{\pi(n-1)}}$ (③)

Applying $\pi_2 = \begin{pmatrix} 1\ 2\ 3\ 4 \dots n \\ 2\ 3\ 1\ 4 \dots n \end{pmatrix}$ and $\pi_3 = \begin{pmatrix} 1\ 2\ 3\ 4 \dots n \\ 3\ 1\ 2\ 4 \dots n \end{pmatrix}$ by same way,

$2S = \sum_{\pi \in S_n} \frac{1}{x_{\pi(2)} x_{\pi(3)}} \frac{1}{x_{\pi(2)} + x_{\pi(3)} + x_{\pi(1)}} \cdots \frac{1}{x_{\pi(1)} + x_{\pi(2)} + \dots + x_{\pi(n-1)}}$ (④)

$2S = \sum_{\pi \in S_n} \frac{1}{x_{\pi(3)} x_{\pi(1)}} \frac{1}{x_{\pi(3)} + x_{\pi(1)} + x_{\pi(2)}} \cdots \frac{1}{x_{\pi(1)} + x_{\pi(2)} + \dots + x_{\pi(n-1)}}$ (⑤)

by adding equation ③, ④ and ⑤,

$3 \times 2S = \sum_{\pi \in S_n} \left(\frac{1}{x_{\pi(1)} x_{\pi(2)}} + \frac{1}{x_{\pi(2)} x_{\pi(3)}} + \frac{1}{x_{\pi(3)} x_{\pi(1)}} \right) \frac{1}{x_{\pi(1)} + x_{\pi(2)} + x_{\pi(3)}} \cdots$

$= \sum_{\pi \in S_n} \frac{1}{x_{\pi(1)} x_{\pi(2)} x_{\pi(3)}} \frac{1}{x_{\pi(1)} + x_{\pi(2)} + x_{\pi(3)} + x_{\pi(4)}} \cdots$

By repeating same method, we finally get

$(n-1)! S = \sum_{\pi \in S_n} \frac{1}{x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n-1)}}$

$$n!S=\sum_{\pi\in S_n}\frac{x_{\pi(1)}+x_{\pi(2)}+...+x_{\pi(n)}}{x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n-1)}}=0$$ (since $x_1+x_2+...+x_n=0$)

$\therefore S=0.$

Therefore, the only possible value of given sum is 0.