Problem of the Week 2012-1

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Problem: Compute $\tan^{-1}(1) - \tan^{-1}(1/3) + \tan^{-1}(1/5) - \tan^{-1}(1/7) + \ldots$.

Solution:

Using the angle subtraction formula for the tangent function, given by

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta},$$

we obtain

$$\tan^{-1}\left(\frac{1}{n}\right) - \tan^{-1}\left(\frac{1}{n+2}\right) = \tan^{-1}\left[\frac{1/n - 1/(n+2)}{1 + 1/n(n+2)}\right] = \tan^{-1}\left[\frac{2}{(n+1)^2}\right].$$

Hence the series considered here can be turned into the following form:

$$\begin{aligned} &\tan^{-1}(1) - \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{7}\right) + \ldots \\ &= \sum_{m=0}^{\infty}\left[\tan^{-1}\left(\frac{1}{4m+1}\right) - \tan^{-1}\left(\frac{1}{4m+3}\right)\right] \\ &= \sum_{m=0}^{\infty}\tan^{-1}\left[\frac{1}{2(2m+1)^2}\right] \end{aligned} \tag{1}$$

Now let us represent the inverse tangent function as follows:

$$\tan^{-1} x = \mathrm{Im}\big[\ln(1+ix)\big], \tag{2}$$

where we have chosen the convention for the natural logarithm $\big[\ln z \equiv \ln|z| + i\arg(z)\big]$ such that the branch cut is placed at $(-\infty, 0]$ and $-\pi < \arg(z) < \pi$.
Then,

$$\begin{aligned} \sum_{m=0}^{\infty}\tan^{-1}\left[\frac{1}{2(2m+1)^2}\right] &= \mathrm{Im}\left[\sum_{m=0}^{\infty}\ln\left(1 + \frac{i}{2(2m+1)^2}\right)\right] \\ &= \mathrm{Im}\left\{\ln\left[\prod_{m=0}^{\infty}\left(1 + \frac{i}{2(2m+1)^2}\right)\right]\right\} \\ &= \mathrm{Im}\left\{\ln\left[\cos\frac{\pi}{4}(1-i)\right]\right\} \\ &= \arg\left[\cos\frac{\pi}{4}(1-i)\right]. \end{aligned} \tag{3}$$

To get the third equality on the above, we have used the infinite product formula for the cosine function:

$$\cos x = \prod_{m=0}^{\infty} \left(1 - \frac{4x^2}{(2m+1)^2\pi^2}\right) .$$

We then have

$$\begin{aligned} \cos\frac{\pi}{4}(1-i) &= \cos\frac{\pi}{4}\cos\frac{\pi i}{4} + \sin\frac{\pi}{4}\sin\frac{\pi i}{4} \\ &= \frac{1}{\sqrt{2}}\left(\cosh\frac{\pi}{4} + i\sinh\frac{\pi}{4}\right) , \end{aligned} \tag{4}$$

and therefore,

$$\arg\left[\cos\frac{\pi}{4}(1-i)\right] = \tan^{-1}\left(\tanh\frac{\pi}{4}\right) .$$