Solution of KAIST POW 2012-1

Hun-Min, Park

Daejeon Science High School(1st grade)

mathematicianscott@gmail.com

Theorem.
Let n be a positive integer and let S_n be the set of all permutations on $\{1, 2, \cdots, n\}$. Assume $x_1 + x_2 + \cdots + x_n = 0$ and $\sum_{i \in A} x_i \neq 0$ for all nonempty proper subsets A of $\{1, 2, \cdots, n\}$. Then

$$\sum_{\pi \in S_n} \frac{1}{x_{\pi(1)}} \frac{1}{x_{\pi(1)} + x_{\pi(2)}} \cdots \frac{1}{x_{\pi(1)} + \cdots + x_{\pi(n-1)}} = 0$$

Proof.
For simplicity, let

$$T := \sum_{\pi \in S_n} \frac{1}{x_{\pi(1)}} \frac{1}{x_{\pi(1)} + x_{\pi(2)}} \cdots \frac{1}{x_{\pi(1)} + \cdots + x_{\pi(n-1)}}$$

and

$$\begin{aligned} A_1 &:= \{(\{\pi(1)\}, \pi(2), \pi(3), \cdots, \pi(n)) | \pi \in S_n\} \\ A_2 &:= \{(\{\pi(1), \pi(2)\}, \pi(3), \cdots, \pi(n)) | \pi \in S_n\} \\ A_3 &:= \{(\{\pi(1), \pi(2), \pi(3)\}, \pi(4), \cdots, \pi(n)) | \pi \in S_n\} \\ &\vdots \\ A_{n-1} &:= \{(\{\pi(1), \pi(2), \cdots, \pi(n-1)\}, \pi(n)) | \pi \in S_n\} \end{aligned}$$

To compute the value of the sum T, we confirm the following lemma first;

[Lemma]
Let $t_1, t_2, \cdots, t_n$ n given numbers($n - 1 \geq 1$). Assume $\sum_{i \in A} t_i \neq 0$ for all nonempty proper subsets A of $\{1, 2, \cdots, n\}$. Then

$$\sum_{\pi \in S_n} \frac{1}{t_{\pi(1)}} \frac{1}{t_{\pi(1)} + t_{\pi(2)}} \cdots \frac{1}{t_{\pi(1)} + \cdots + t_{\pi(n-1)}} = \frac{t_1 + \cdots + t_n}{t_1 \cdots t_n}$$

[*Proof of lemma.*]
Let

$$S := \sum_{\pi \in S_n} \frac{1}{t_{\pi(1)}} \frac{1}{t_{\pi(1)} + t_{\pi(2)}} \cdots \frac{1}{t_{\pi(1)} + \cdots + t_{\pi(n-1)}}$$

Then

$$S = \sum_{(\{a_1\},a_2,\cdots,a_n)\in A_1} \frac{1}{t_{a_1}} \frac{1}{t_{a_1}+t_{a_2}} \cdots \frac{1}{t_{a_1}+\cdots+t_{a_{n-1}}}$$

$$= \sum_{(\{a_1,a_2\},a_3,\cdots,a_n)\in A_2} \left(\frac{1}{t_{a_1}}+\frac{1}{t_{a_2}}\right) \frac{1}{t_{a_1}+t_{a_2}} \frac{1}{t_{a_1}+t_{a_2}+t_{a_3}} \cdots \frac{1}{t_{a_1}+\cdots+t_{a_{n-1}}}$$

$$= \sum_{(\{a_1,a_2\},a_3,\cdots,a_n)\in A_2} \frac{1}{t_{a_1}t_{a_2}} \frac{1}{t_{a_1}+t_{a_2}+t_{a_3}} \cdots \frac{1}{t_{a_1}+\cdots+t_{a_{n-1}}}$$

$$= \sum_{(\{a_1,a_2,a_3\},a_4,\cdots,a_n)\in A_3} \left(\frac{1}{t_{a_1}t_{a_2}}+\frac{1}{t_{a_2}t_{a_3}}+\frac{1}{t_{a_3}t_{a_1}}\right) \frac{1}{t_{a_1}+t_{a_2}+t_{a_3}} \cdots \frac{1}{t_{a_1}+\cdots+t_{a_{n-1}}}$$

$$= \sum_{(\{a_1,a_2,a_3\},a_4,\cdots,a_n)\in A_3} \frac{1}{t_{a_1}t_{a_2}t_{a_3}} \frac{1}{t_{a_1}+t_{a_2}+t_{a_3}+t_{a_4}} \cdots \frac{1}{t_{a_1}+\cdots+t_{a_{n-1}}}$$

$$= \cdots$$

$$\vdots$$

$$= \sum_{(\{a_1,\cdots,a_k\},a_{k+1},\cdots,a_n)\in A_k} \underbrace{\left(\frac{1}{t_{a_1}\cdots t_{a_k}}+\cdots+\frac{1}{t_{a_2}\cdots t_{a_{k+1}}}\right)}_{k+1} \frac{1}{t_{a_1}+\cdots+t_{a_{k+1}}} \cdots \frac{1}{t_{a_1}+\cdots+t_{a_{n-1}}}$$

$$= \sum_{(\{a_1,\cdots,a_k\},a_{k+1},\cdots,a_n)\in A_k} \frac{1}{t_{a_1}\cdots t_{a_{k+1}}} \frac{1}{t_{a_1}+\cdots+t_{a_{k+2}}} \cdots \frac{1}{t_{a_1}+\cdots+t_{a_{n-1}}}$$

$$= \cdots$$

$$\vdots$$

$$= \sum_{(\{a_1,\cdots,a_{n-1}\},a_n)\in A_{n-1}} \frac{1}{t_{a_1}t_{a_2}\cdots t_{a_{n-1}}}$$

(We've used the identity $(1/(j_1+\cdots+j_{m+1}))\cdot \sum_{sym}(1/(j_1 j_2\cdots j_m))$
$= \underbrace{(1/(j_1\cdots j_m)+\cdots+1/(j_2\cdots j_{m+1}))}_{m+1} \cdot (1/(j_1+\cdots+j_{m+1})) = 1/(j_1\cdots j_{m+1}))$

Hence

$$S = \sum_{(\{a_1,\cdots,a_{n-1}\},a_n)\in A_{n-1}} \frac{1}{t_{a_1}t_{a_2}\cdots t_{a_{n-1}}} = \frac{t_1+\cdots+t_n}{t_1\cdots t_n}$$

and our lemma is proved. $\triangle$

Therefore

$$T = \sum_{\pi\in S_n} \frac{1}{x_{\pi(1)}} \frac{1}{x_{\pi(1)}+x_{\pi(2)}} \cdots \frac{1}{x_{\pi(1)}+\cdots+x_{\pi(n-1)}} = \frac{x_1+\cdots+x_n}{x_1\cdots x_n} = 0$$

$(\because x_1+\cdots+x_n=0)$
So the possible value of T is only 0. $\square$