Proof. For any n numbers $x_1, x_2, \cdots, x_n$ satisfying $\sum_{i\in A} x_i \neq 0$ for all nonempty proper subset A of $\{1, 2, \cdots, n\}$, define the function f_n:

$$f_n(x_1, x_2, \cdots, x_n) = \sum_{\pi\in S_n} \frac{1}{x_{\pi(1)}} \frac{1}{x_{\pi(1)} + x_{\pi(2)}} \cdots \frac{1}{x_{\pi(1)} + \cdots + x_{\pi(n-1)}}$$

Claim.

$$f_n(x_1, x_2, \cdots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{x_1 x_2 \cdots x_n}$$

Proof. Use induction on n: It is obvious when $n = 1, 2$.
Assume the statement holds for $n = k-1$. Let $x_1, \cdots, x_k$ be k numbers satisfying the condition. And define S_k^i be the subset of S_k such that for all $\pi \in S_k^i$, $\pi(k) = x_i$. Then,

$$\begin{aligned}
f_k(x_1, x_2, \cdots, x_k) &= \sum_{\pi\in S_k} \frac{1}{x_{\pi(1)}} \cdots \frac{1}{x_{\pi(1)} + \cdots + x_{\pi(k-1)}} \\
&= \sum_{i=1}^{k} \sum_{\pi\in S_k^i} \frac{1}{x_{\pi(1)}} \cdots \frac{1}{x_{\pi(1)} + \cdots + x_{\pi(k-1)}} \\
&= \sum_{i=1}^{k} \sum_{\pi\in S_k^i} \frac{1}{x_{\pi(1)}} \cdots \frac{1}{x_{\pi(1)} + \cdots + x_{\pi(k-2)}} \frac{1}{x_{\pi(1)} + \cdots + x_{\pi(k-1)}} \\
&= \sum_{i=1}^{k} \sum_{\pi\in S_k^i} \frac{1}{x_{\pi(1)}} \cdots \frac{1}{x_{\pi(1)} + \cdots + x_{\pi(k-2)}} \frac{1}{x_1 + \cdots + x_k - x_i} \\
&= \sum_{i=1}^{k} \frac{1}{x_1 + \cdots + x_k - x_i} \sum_{\pi\in S_k^i} \frac{1}{x_{\pi(1)}} \cdots \frac{1}{x_{\pi(1)} + \cdots + x_{\pi(k-2)}} \\
&= \sum_{i=1}^{k} \frac{1}{x_1 + \cdots + x_k - x_i} f_{k-1}(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k)
\end{aligned}$$

The last equality holds because we can think one-to-one correspondence between S_k^i and S_{k-1}. Using the inductive hypothesis, It can be shown that

$$\begin{aligned}
f_k(x_1, x_2, \cdots, x_k) &= \sum_{i=1}^{k} \frac{1}{x_1 + \cdots + x_k - x_i} f_{k-1}(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k) \\
&= \sum_{i=1}^{k} \frac{1}{x_1 + \cdots + x_k - x_i} \frac{x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_k}{x_1 \cdots x_{i-1} x_{i+1} \cdots x_k} \\
&= \sum_{i=1}^{k} \frac{1}{x_1 + \cdots + x_k - x_i} \frac{x_1 + \cdots + x_k - x_i}{x_1 \cdots x_{i-1} x_{i+1} \cdots x_k} \\
&= \sum_{i=1}^{k} \frac{1}{x_1 \cdots x_{i-1} x_{i+1} \cdots x_k} = \sum_{i=1}^{k} \frac{x_i}{x_1 \cdots x_k} = \frac{x_1 + \cdots + x_k}{x_1 \cdots x_k}
\end{aligned}$$

Therefore the claim has been proved by mathematical induction. □

So, by the claim, when $x_1 + \cdots + x_n = 0$, zero is the only possible value of given equation. □