

Proof. It is enough to show that $\forall n \in \mathbf{N}, \forall c_i \in \mathbf{R} (i = 1, \dots, n)$ satisfying $\sum_{i=1}^n c_i = 0$, there exist $m \in \{1, \dots, n\}$ such that

$$\sum_{k=0}^s c_{m+k} \geq 0 \quad \forall s = 0, 1, \dots, n-2$$

(When $m > n$, treat $c_m = c_{m-n}$). (Why: Let n be the number of stations. Label them as $1, \dots, n$, in clockwise direction. Let a_i be the amount of money hidden in station i , b_i be the cost to move from station i to $i+1$. Then finding the station able to start (in clockwise direction) means the same with above statement, when we define $c_i = a_i - b_i$.)

We use induction on n to prove this statement; When $n = 2$, since $c_1 + c_2 = 0$, either of $i = 1, 2$ satisfies $c_i \geq 0$. Then $m = i$ satisfies the condition above.

Assume the statement holds for $n = l$. Now let $c_i (i = 1, \dots, l+1)$ be the real numbers satisfying $\sum_{i=1}^{l+1} c_i = 0$. Then obviously one of them is not negative. WLOG let the one c_i . Now define $c'_i (i = 1, \dots, l)$ as follows:

$$c'_1 = c_1 + c_2 \text{ and } c'_i = c_{i+1} \text{ for } i = 2, \dots, l$$

Then it is easy to check that $\sum_{i=1}^l c'_i = 0$. So from induction hypothesis, there exist $m' \in \{1, \dots, l\}$ such that $\sum_{k=0}^s c'_{m'+k} \geq 0$ for $s = 0, 1, \dots, l-2$. Now set $m := 1$ if $m' = 1$ and $m := m' + 1$ otherwise. Then as s varies in the equation above, we get $\sum_{k=0}^s c_{m+k} \geq 0$ for $s = 0, 1, \dots, l-1$ except when $c_{m+s} = c_1$. In other words, all the sum $c_m + \dots + c_{m+s}$ is proved to be non-negative, except when it ends with c_1 . But since we already know that $c_1 \geq 0$, this one also can be proved easily. Therefore all the sum is non-negative, so the statement holds for $n = l+1$.

In conclusion, by induction on n , we proved the foremost statement which means that we can choose a station to start for a roundtrip without any money matters. Thus, there is always a station to start for a roundtrip of Subway Line 2, regardless of number of station, amount of money hidden or payment for each segment. \square