Proof. Let $x = \sum x_i/2^i$ be a binary notation of x. Now we want to compute $d(2^{k-1}x)$ from this notation. It is obvious that $d(x) = d(x - [x])$ for any x. Then,

$$\begin{aligned}
d(2^{k-1}x) &= d(\sum_{i=-\infty}^{\infty} \frac{x_i}{2^{i-k+1}}) = d([\sum_{i=-\infty}^{\infty} \frac{x_i}{2^{i-k+1}}]) = d(\sum_{i=k}^{\infty} \frac{x_i}{2^{i-k+1}}) \\
&= \begin{cases} (\sum_{i=k}^{\infty} \frac{x_i}{2^{i-k+1}})^2 & \text{if } x_k = 0 \\ (1 - \sum_{i=k}^{\infty} \frac{x_i}{2^{i-k+1}})^2 & \text{if } x_k = 1 \end{cases} \\
&= (1 - x_k)(\sum_{i=k}^{\infty} \frac{x_i}{2^{i-k+1}})^2 + x_k(1 - \sum_{i=k}^{\infty} \frac{x_i}{2^{i-k+1}})^2 \\
&= (\sum_{i=k}^{\infty} \frac{x_i}{2^{i-k+1}})^2 - 2x_k(\sum_{i=k}^{\infty} \frac{x_i}{2^{i-k+1}}) + x_k
\end{aligned}$$

We should evaluate $\cdots + 4d(x/4) + 2d(x/2) + d(x) + d(2x)/2 + d(4x)/4 + \cdots$, which is equal to

$$\sum_{k=-\infty}^{\infty} \frac{d(2^{k-1}x)}{2^{k-1}} = \sum_{k=-\infty}^{\infty} \frac{1}{2^{k-1}} \left\{ (\sum_{i=k}^{\infty} \frac{x_i}{2^{i-k+1}})^2 - 2x_k(\sum_{i=k}^{\infty} \frac{x_i}{2^{i-k+1}}) + x_k \right\}$$

In the above expansion, coefficient of x_n^2 is

$$\sum_{k=-\infty}^{n} \frac{1}{2^{k-1}} (\frac{1}{2^{n-k+1}})^2 - \frac{1}{2^{n-1}} = \sum_{k=-\infty}^{n} \frac{1}{2^{2n-k+1}} - \frac{1}{2^{n-1}} = \sum_{k=0}^{\infty} \frac{1}{2^{n+k+1}} - \frac{1}{2^{n-1}} = -\frac{1}{2^n}$$

and coefficient of x_n is $\frac{1}{2^{n-1}}$. And lastly, for any $n, m \in \mathbf{Z} (n < m)$, coefficient of $x_n x_m$ is

$$\sum_{k=-\infty}^{n} \frac{1}{2^{k-1}} 2 \frac{1}{2^{n-k+1}} \frac{1}{2^{m-k+1}} - \frac{1}{2^{n-1}} \frac{2}{2^{m-n+1}} = \sum_{k=-\infty}^{n} \frac{1}{2^{n+m-k}} - \frac{1}{2^{m-1}} = 0$$

By substituting all coefficients, we obtain

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \frac{d(2^{k-1}x)}{2^{k-1}} &= \sum_{k=-\infty}^{\infty} \frac{1}{2^{k-1}} \left\{ (\sum_{i=k}^{\infty} \frac{x_i}{2^{i-k+1}})^2 - 2x_k(\sum_{i=k}^{\infty} \frac{x_i}{2^{i-k+1}}) + x_k \right\} \\
&= \sum_{n=-\infty}^{\infty} -\frac{1}{2^n} x_n^2 + \frac{1}{2^{n-1}} x_n \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{2^n} x_n \quad (x_n^2 = x_n \text{ for } x_n = 0, 1) = x
\end{aligned}$$

Answer: $\cdots + 4d(x/4) + 2d(x/2) + d(x) + d(2x)/2 + d(4x)/4 + \cdots = x$

□