POW 2011-14 Invertible matrices

20100

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Let $M = \mathrm{diag}(d_1, \cdots, d_n) = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ where $d_i > 0$

We know that $M-J$ is invertible $\Leftrightarrow$ $(M-J)x = 0$ iff $x = 0 \, (\in \mathbb{R}^n)$.

For "not invertible" $M-J$, if $tr(M) \geq f(n)$ and equality holds for some M, for some function f of n,
then for any M s.t. $tr(M) < f(n)$ $M-J$ is invertible and f is the largest upper bound for $tr(M)$ to $M-J$ to be invertible. ····· $(\star)$

Claim 1 For $M-J$ not invertible, $\exists\, x \in \mathbb{R}^n$ s.t. $(M-J)x = 0$ and $\sum_{i=1}^n x_i \neq 0$.

pf) Suppose there does NOT exist such $x \in \mathbb{R}^n$.
Then, $\forall x \neq 0$ s.t. $(M-J)x = 0$, $\sum_{i=1}^n x_i = 0$.
However $Jx = \begin{pmatrix} \sum_{i=1}^n x_i \\ \vdots \\ \sum_{i=1}^n x_i \end{pmatrix} = 0$, thus $Mx = \begin{pmatrix} d_1 x_1 \\ \vdots \\ d_n x_n \end{pmatrix} = 0$.

$\Rightarrow x = 0$ ($\because d_i > 0$ for each $i = 1, \cdots, n$), contradiction // ($\because$ then col(M-J)=0)

Therefore $\exists\, x \in \mathbb{R}^n$ s.t. $(M-J)x = 0$ and $\sum_{i=1}^n x_i \neq 0$ /// ···· $(*)$

Now, let $M-J$ be not invertible. Then $\exists\, x \in \mathbb{R}^n$ satisfies $(*)$, by claim 1.
Then, from $Jx = Mx$, $d_i x_i = \sum_{i=1}^n x_i$ for each $i = 1, \cdots, n$.

Thus, $1 = \frac{\sum x_i}{\sum x_i} = \sum_{j=1}^n \frac{x_j}{\sum_{i=1}^n x_i} = \sum_{j=1}^n \frac{1}{d_j} \geq n \cdot \sqrt[n]{\frac{1}{d_1 \cdots d_n}}$ (산술-기하 부등식)

$\Rightarrow n \leq \sqrt[n]{d_1 \cdots d_n} \leq \frac{\sum_{i=1}^n d_i}{n} = \frac{tr(M)}{n}$. $\Rightarrow tr(M) \geq n^2$. (산술-기하 부등식)
(Equality holds for $d_1 = \cdots = d_n = n$)

(Note that for $d_1 = d_2 = \cdots = d_n = n$, $x = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$,
$(M-J)x = 0$, Thus $M-J$ is not invertible.)

Therefore $f(n) = n^2$ by $(\star)$ ■