Two matrices

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POW2011-15. *Let n be a positive integer. Let $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$. Suppose that A, B are two complex square matrices such that $AB = \omega BA$. Prove that $(A+B)^n = A^n + B^n$.*

Solution. Let's prove more general result: $(pA + qB)^n = p^n A^n + q^n B^n$ for any complex numbers p, q. ($p = q = 1$ in the original problem.)

Remark that $(pA + qB)^n = \sum_{k=0}^{n} c_k p^k q^{n-k} A^k B^{n-k}$ for some constants c_i with $c_0 = c_n = 1$, which do not depend on the choice of A, B whenever $AB = \omega BA$. Let's choose specific matrices,

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n}, B = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & \omega & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \omega^{n-2} & 0 \\ 0 & 0 & \cdots & 0 & \omega^{n-1} \end{pmatrix}_{n\times n},$$

thus A is the elementary row matrix which inserts the top row to the bottom, or the elementary column matrix which inserts the rightmost column to the leftmost. It is clear that $AB = \omega BA$.

On the other hand, the characteristic polynomial of $pA + qB$ can be deduced from the bottom row as:

$$\begin{aligned} \rho(\lambda) &= \det\left(\lambda I - (pA + qB)\right) \\ &= (-1)^{n-1}(-p)(-p)^{n-1} + (\lambda - q\omega^{n-1})(\lambda - q)\cdots(\lambda - q\omega^{n-2}) \\ &= -p^n + (\lambda^n - q^n) = \lambda^n - (p^n + q^n). \end{aligned}$$

By the Caley-Hamilton theorem, $\rho(pA + qB) = 0$, hence $(pA + qB)^n = (p^n + q^n)I = (pA)^n + (qB)^n$. This implies that $c_i = 0$ for all $1 \le i \le n-1$ as we desired. □