Proof. It is clear that $(A+B)^m = \sum_{k=0}^{m} c_k^m B^k A^{m-k}$ where the scalars c_k^m only depend on k and m. By some computing c_k^m for small numbers, we can guess that

$$c_k^m = \frac{\phi_m}{\phi_k \phi_{m-k}},$$

where ϕ_k's are given by

$$\phi_k = \prod_{i=1}^{k}(1+\cdots+\omega^{i-1})$$

(Assume $\phi_0 = 1$)

Claim. When c_k^m's are defined from the equation $(A+B)^m = \sum_{k=0}^{m} c_k^m B^k A^{m-k}$, for all m, we get $c_0^m = c_m^m = 1$ and for $k = 1, \cdots, m-1$,

$$c_k^m = \frac{\phi_m}{\phi_k \phi_{m-k}}$$

Proof. First of all, we can easily compute that $c_0^m = c_m^m = 1$ for all m. So it is enough to compute for $k = 1, \cdots, m-1$.
By computing $(A+B)^{m+1} = (A+B)\sum_{k=0}^{m} c_k^m B^k A^{m-k}$, we get

$$c_k^{m+1} = \omega^k c_k^m + c_{k-1}^m$$

for $k = 1, \cdots, m$. Now use induction on m:
$m = 0, 1$: Obvious because we showed all the possible cases already (either $k = 0$ or m)
Assume the equation holds on $m = l$. Then for all $k = 1, \cdots, l$,

$$\begin{aligned}
c_k^{l+1} &= \omega^k c_k^l + c_{k-1}^l \\
&= \omega^k \frac{\phi_l}{\phi_k \phi_{l-k}} + \frac{\phi_l}{\phi_{k-1}\phi_{l-k+1}} \\
&= \frac{\phi_l}{\phi_k \phi_{l-k+1}}\left(\omega^k \frac{\phi_{l-k+1}}{\phi_{l-k}} + \frac{\phi_k}{\phi_{k-1}}\right) \\
&= \frac{\phi_l}{\phi_k \phi_{l-k+1}}\{\omega^k(1+\cdots+\omega^{l-k}) + (1+\cdots+\omega^{k-1})\} \\
&= \frac{\phi_l}{\phi_k \phi_{l-k+1}}(1+\cdots+\omega^l) = \frac{\phi_{l+1}}{\phi_k \phi_{l-k+1}}
\end{aligned}$$

Therefore, by induction, claim holds for all m and $k = 0, 1, \cdots, m$. □

We know that $\omega^i - 1 \neq 0$ for $i = 1, \cdots, n-1$. Thus $1+\omega+\cdots+\omega^{i-1} \neq 0$ for $i = 1, \cdots, n-1$. It means that $\phi_i \neq 0$ for $i = 1, \cdots, n-1$. But $\phi_n = 0$, since $(\omega-1)(1+\omega+\cdots+\omega^{n-1}) = \omega^{n-1} = 0$ and $\omega \neq 1$. Therefore, for $k = 1, \cdots, n-1$, by using claim,

$$c_k^n = \frac{\phi_n}{\phi_k \phi_{n-k}} = \frac{0}{\phi_k \phi_{n-k}} = 0$$

and $c_0^n = c_n^n = 1$. Consequently, we get

$$(A+B)^n = \sum_{k=0}^{n} c_k^n B^k A^{n-k} = A^n + B^n$$

□