Sums of Partial Sums

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POW2011-13. *Let $a_1, a_2, \cdots$ be a sequence of non-negative real numbers less than or equal to 1. Let $S_n = \sum_{i=1}^n a_i$ and $T_n = \sum_{i=1}^n S_i$. Prove or disprove that $\sum_{n=1}^\infty \frac{a_n}{T_n}$ converges. (Assume $a_1 > 0$.)*

Solution.

Claim. *$T_n \geq S_{n-1}S_n/2$ for $n > 1$.*

Proof. Since $0 \leq a_i \leq 1$, it follows from

$$\begin{aligned}
T_n &= na_1 + (n-1)a_2 + \cdots + a_n \\
&\geq (a_1 + \cdots + a_n)a_1 + (a_2 + \cdots + a_n)a_2 + \cdots + a_n \\
&= \sum_{i=1}^n a_i^2 + \sum_{i<j} a_i a_j = \left(\sum_{i=1}^n a_i\right)^2 - \sum_{i<j} a_i a_j \\
&\geq S_n^2 - \sum_{i<j} a_i = S_n^2 - ((n-1)a_1 + (n-2)a_2 + \cdots + a_{n-1}) \\
&= S_n^2 - T_{n-1} \geq S_{n-1}S_n - T_n.
\end{aligned}$$

□

By the claim with $S_i > 0$,

$$\begin{aligned}
\sum_{n=1}^\infty \frac{a_n}{T_n} &\leq 1 + \sum_{n=2}^\infty \frac{2a_n}{S_{n-1}S_n} \\
&= 1 + \sum_{n=2}^\infty 2\left(\frac{1}{S_{n-1}} - \frac{1}{S_n}\right) \\
&= 1 + \frac{2}{S_1} = 1 + \frac{2}{a_1}.
\end{aligned}$$

Thus, the given bounded (positive-term) series converges. □

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If S_n is unbounded, then 1/S_n converges to 0 and so the equality holds here.
However, if S_n is bounded, then 1/S_n will converge to 1/c for some constant c>0. So the equlaity does not hold in general.