Distinct prime factors

Minjae Park

POW2011-9. *Prove that there is a constant $c > 1$ such that if $n > c^k$ for positive integers n and k, then the number of distinct prime factors of $\binom{n}{k}$ is at least k.*

Solution. Let us write $[a, b] = \{a, a+1, \cdots, b-1, b\}$ for $b > a$. Let $\omega(n)$ be the number of distinct prime factors of n, and $v_p(n)$ be the number satisfying $p^{v_p(n)} \parallel n$ for a prime p. Let $\pi(n)$ be the prime counting function.

Theorem 1. *If $n \geq k! + k$, then $\omega(\binom{n}{k}) \geq k$.*

Proof. Let $n \geq k! + k$, and $\displaystyle\binom{n}{k} = \prod_{j=1}^{m} p_j^{e_j}$ be the prime factorization.

For any p_j, let $a_i = \#\{s \in [n-k+1, n] \text{ s.t. } p_j^i \mid s\}$. Let α be the largest i so that $a_i \neq 0$. Similarly, let $b_i = \#\{s \in [1, k] \text{ s.t. } p_j^i \mid s\}$. Note that $\#[n-k+1, n] = \#[1, k] = k$, so $a_i \leq b_i + 1$ for all i. Also, $a_i = 0$ for any $i > \alpha$. Then,

$$\begin{aligned} e_j = v_{p_j}(\binom{n}{k}) &= v_{p_j}(n(n-1)\cdots(n-k+1)) - v_{p_j}(k!) \\ &= \sum_{i \geq 1} a_i - \sum_{i \geq 1} b_i = \sum_{i \geq 1}(a_i - b_i) \leq \sum_{i=1}^{\alpha} 1 = \alpha \end{aligned}$$

hold. This implies that $\exists s \in [n-k+1, n]$ so that $p_j^{e_j} \mid s$.

Suppose that $m = \omega\left(\binom{n}{k}\right) < k$. Note that $\displaystyle\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \prod_{j=1}^{m} p_j^{e_j}$, and at most $m < k$ terms in the numerator of the left hand side divide all terms of the right hand side. Thus, there should be $s \in [n-k+1, n]$ which divides $k!$. However, this is impossible because $n \geq k! + k$, so $s > k!$. Therefore, $\omega\left(\binom{n}{k}\right) \geq k$. □

Lemma 2. *For any $\epsilon > 0$, there exists k_0 such that if $k > k_0$ and $n > (e+\epsilon)^k$, then there is a prime p_i satisfying $p_i^{e_i} \parallel n-i$ and $p_i^{e_i} > k$ for every i with $0 \le i < k$.*

Proof. Assume that there is no such prime for some $n-i$ with $0 \le i < k$. Let $n - i = \prod_{j=1}^{m} p_j^{e_j}$ be the prime factorization. Since each $p_j^{e_j} \le k$, we obtain

$$n - i \le k^m \le k^{\pi(k)} = e^{\pi(k)\log(k)} = e^{(1+o(1))k}$$

by the prime number theorem. This is a contradiction for sufficiently large k, for $n > (e+\epsilon)^k$. □

Theorem 3 (P. Erdös, H. Gupta, S. P. Khare, 1976)**.** *For any $\epsilon > 0$, there exists k_0 such that if $k > k_0$ and $n > (e+\epsilon)^k$, then $\omega\left(\binom{n}{k}\right) \ge k$.*

Proof. Let $\epsilon > 0$ be given. By the lemma 2, there exists k_0 such that if $k > k_0$ and $n > (e+\epsilon)^k$, then there is a prime p_i satisfying $p_i^{e_i} \parallel n-i$ and $p_i^{e_i} > k$ for every i with $0 \le i < k$. Note that $p_i \mid \binom{n}{k}$ for all i with $0 \le i < k$, and $p_i \ne p_j$ if $i \ne j$ because $p_i^{e_i} > k$. Therefore, the theorem 3 is immediately obtained. □

By the theorem 3, there exists k_0 such that if $k > k_0$ and $n > 3^k$, then $\omega\left(\binom{n}{k}\right) \ge k$. Let $c = \max\{3, k_0! + k_0\} > 1$. If $k \le k_0$ and $n > c^k$, then $n > c^k \ge c \ge k_0! + k_0 \ge k! + k$, so $\omega\left(\binom{n}{k}\right) \ge k$ by the theorem 1. If $k > k_0$ and $n > c^k$, then $n > c^k \ge 3^k$, so $\omega\left(\binom{n}{k}\right) \ge k$. Consequently, the original problem is proved. □