Factorial

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POW2011-7. *Let $f(n)$ be the largest integer k such that $n!$ is divisible by n^k. Prove that*

$$\lim_{n\to\infty} \frac{\log n \max_{2\le i\le n} f(i)}{n \log\log n} = 1.$$

Solution. Let $n = \prod_{j=1}^m {p_j}^{e_j}$ be the prime factorization. Since $n^{f(n)} \mid n!$,

$$f(n)e_j \le \sum_{i\ge1} \left[\frac{n}{p_j^i}\right] < \sum_{i\ge1} \frac{n}{p_j^i} = \frac{n}{p_j - 1}$$

$$e_j \log p_j \le \frac{n \log p_j}{f(n)(p_j-1)}$$

for all j. Thus,

$$\log n = \sum_{j=1}^m e_j \log p_j \le \frac{n}{f(n)} \sum_{j=1}^m \frac{\log p_j}{p_j - 1}$$

holds. Let $q_1 < q_2 < \cdots$ be all prime numbers. Then, $\frac{\log p_j}{p_j-1} < \frac{\log q_j}{q_j-1}$ because $\frac{\log p}{p-1}$ is decreasing. By the prime number theorem and Mertens' theorem, we obtain

$$\sum_{j=1}^m \frac{\log p_j}{p_j-1} \le \sum_{j=1}^m \frac{\log q_j}{q_j-1} = \sum_{\substack{p\le q_m \\ p;prime}} \frac{\log p}{p-1}$$

$$\le \log\left(m \log\left(m\log m\right)\right) + O(1) = \log m\left(1+o(1)\right)$$

so the following is derived from the previous inequalities.

$$f(n) \le \frac{n\log m}{\log n}(1+o(1))$$

Note that $n = \prod_{j=1}^m {p_j}^{e_j} \ge 2^m$, so $\log m \le \log\log n + O(1)$. Therefore,

$$f(n) \le \frac{n \log\log n}{\log n}(1+o(1)) \tag{1}$$

Let s be the number satisfying $(s+1)! \le n < (s+2)!$. By Stirling's approximation, $s! \sim \sqrt{2\pi s}\left(\frac{s}{e}\right)^s$, so $s \sim \frac{\log n}{W(\log n)}$ where $W(n)$ is the Lambert W function. Since $W(n) \sim \log n$, we have $s \sim \frac{\log n}{\log\log n}$. Let q be the largest prime with $q \le \sqrt[3]{\frac{n}{s!}}$. Let $t = q^3 s! \le n$. Note that $q \sim \sqrt[3]{\frac{n}{s!}}$ because $\frac{n}{s!} \ge s+1 \to \infty$ as $n \to \infty$, so $t \sim n$.

Let $v_p(n)$ be the number satisfying $p^{v_p(n)} \parallel n$ for a prime p. Then,

$$v_p(t!) = \sum_{i\ge 1}\left[\frac{t}{p^i}\right] \ge \frac{t}{s}\sum_{i\ge 1}\left[\frac{s}{p^i}\right] = \frac{t}{s}v_p(s!)$$

holds for any prime p, i.e. $(s!)^{t/s} \mid t!$. For the exponents of q, we have

$$v_q(t) = v_q(q^3 s!) = 3 + \sum_{i\ge 1}\left[\frac{s}{q^i}\right] < 3 + \frac{s}{q-1}$$

$$v_q(t!) = \sum_{i\ge 1}\left[\frac{t}{q^i}\right] \ge \left[\frac{t}{q}\right] > \frac{t}{q} - 1.$$

Therefore,

$$f(t) \ge \min\left\{\frac{t}{s}, \frac{v_q(t!)}{v_q(t)}\right\} \ge \min\left\{\frac{t}{s}, \frac{\frac{t}{q}-1}{\frac{s}{q-1}+3}\right\} \sim \frac{t}{s}$$

since $q \le \sqrt[3]{\frac{n}{s!}} < \sqrt[3]{\frac{(s+2)!}{s!}} = \sqrt[3]{(s+1)(s+2)} = o(s)$. Consequently,

$$\max_{2\le i\le n} f(i) \ge f(t) \ge (1+o(1))\frac{t}{s} = (1+o(1))\frac{n}{\frac{\log n}{\log\log n}}$$
$$= (1+o(1))\frac{n\log\log n}{\log n} \qquad (2)$$

and $\max_{2\le i\le n} f(i) \sim \frac{n\log\log n}{\log n}$ from (1) and (2), so

$$\lim_{n\to\infty}\frac{\log n \max_{2\le i\le n} f(i)}{n\log\log n} = 1 \qquad \square$$