Skew-symmetric and symmetric matrices

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POW2011-11. *Prove that for every skew-symmetric matrix A, there are symmetric matrices B and C such that $A = BC - CB$.*

Solution.

Theorem 1. *Every square matrix A is a product of two symmetric matrices.*

Proof. Let $A = Q^{-1}XQ$ be the rational canonical form, i.e. Q is the invertible matrix, and $X = \operatorname{diag}(A_1, A_2, \cdots, A_m)$, where each A_i has a form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 1 & 0 & \cdots & 0 & a_2 \\ 0 & 1 & \cdots & 0 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_n \end{pmatrix}.$$

If X is a product of two symmetric matrices Y and Z, then $B = Q^{-1}Y(Q^T)^{-1}$ and $C = Q^T ZQ$ are also symmetric matrices with $A = BC$. Thus, it is sufficient to prove the theorem only for A_i. Let

$$C_i = \begin{pmatrix} a_2 & a_3 & a_4 & \cdots & a_n & -1 \\ a_3 & a_4 & 0 & \cdots & -1 & 0 \\ a_4 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ a_n & -1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

be a symmetric matrix, then it is invertible since $|C_i| \neq 0$, so C_i^{-1} is also symmetric. In addition, it is easy to check that A_iC_i is symmetric by direct computation. Therefore, A_i is a product of two symmetric matrices A_iC_i and C_i^{-1}, which completes the proof. □

For given skew-symmetric matrix A, there is a square matrix X so that $A = X - X^T$. For example, take a triangular part of A. By Theorem 1, there exist two symmetric matrices B and C with $X = BC$. Therefore, $A = BC - (BC)^T = BC - CB$. □