POW 2011-11 Skew-symmetric and symmetric matrices.
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Prove that for every skew-symmetric matrix A, there are symmetric matrices B and C such that $A=BC-CB$.

Solution : Let $A\in M_{n\times n}$ (any skew-symmetric matrix) be given. Let $B=(i\delta_{ij})$.
(Then B is symmetric) We will show that there is a symmetric matrix C such that $A=BC-CB$.

<u>**STEP1**</u> : Let's define a function $f_B : M_{n\times n} \to M_{n\times n}$ as $f_B(X)=BX-XB$ and let $C_{i,j}$ be a matrix such that (i,j), (j,i) components of $C_{i,j}$ are 1, and the other components of $C_{i,j}$ are 0. (Then f_B is a linear function and C is symmetric)

Then $f_B(C_{i,j})$ be a matrix such that (i,j) component of $f_B(C_{i,j})$ is $i-j$, (j,i) component of $f_B(C_{i,j})$ is $-(i-j)$, and the other components of $f_B(C_{i,j})$ are 0.

<u>**STEP2**</u> : For $1\le i<j\le n$, let $\alpha_{i,j}=\dfrac{a_{ij}}{i-j}$ where $A=(a_{ij})$.

<u>**STEP3**</u> : Let $C=\sum_{1\le i<j\le n}\alpha_{ij}C_{i,j}$.

Since f_B is a linear function, $f_B(C)=f_B\left(\sum_{1\le i<j\le n}\alpha_{ij}C_{i,j}\right)=\sum_{1\le i<j\le n}\alpha_{ij}f(C_{i,j})=A$.

Since $f_B(C)=BC-CB$, We are done.■