POW 2011-8: Geometric Mean

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Let $M = \max_{0 \le x \le 1} f(x) < \infty$ and fix $\epsilon > 0$. Observe that there exists $\delta > 0$ such that

$$\left| f(x) - f\left(\frac{1}{e}\right) \right| < \frac{\epsilon}{2}$$

whenever $|x - 1/e| < \delta$. Define

$$S_n = \{(x_1, \ldots, x_n) \in (0,1)^n : |\sqrt[n]{x_1 x_2 \ldots x_n} - 1/e| \ge \delta\}.$$

Through some calculations, we know that

$$S_n \subset \left\{ (x_1, \ldots, x_n) : \left| \frac{\ln x_1 + \ldots + \ln x_n}{n} - (-1) \right| \ge -\ln(1 - e\delta) \right\} = T_n.$$

Define $Y_i : (0,1) \to (-\infty, 0)$ as random variables such that $Y_i(x) = \ln x$ with uniform distribution on $(0,1)$. Then, $\mathbb{E}(Y_i) = -1$, $\mathbb{E}(Y_i^2) = 2/27 < \infty$, and

$$\mu(T_n) = \mathbb{P}\left(\left| \frac{Y_1 + \ldots + Y_n}{n} - (-1) \right| \ge -\ln(1 - e\delta) \right)$$

where μ is the standard measure on $(0,1)$. Hence, by weak law of large numbers, there exist $N > 0$ with $\mu(T_n) < \epsilon/4M$ for any $n > N$. Notice that

$$\begin{aligned}
&\int_{(0,1)^n} |f(\sqrt[n]{x_1 \ldots x_n}) - f(1/e)| d\mu \\
&= \int_{x \in S_n} |f(\sqrt[n]{x_1 \ldots x_n}) - f(1/e)| d\mu + \int_{x \notin S_n} |f(\sqrt[n]{x_1 \ldots x_n}) - f(1/e)| d\mu \\
&\le 2M\mu(S_n) + \frac{\epsilon}{2}\mu([0,1]^n) \\
&\le 2M\mu(T_n) + \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

for any $n > N$. Thus,

$$\lim_{n \to \infty} \int_{[0,1]^n} |f(\sqrt[n]{x_1 \ldots x_n}) - f(1/e)| d\mu = 0$$

and we are done.