2011#6 Let $a_1 \le \cdots \le a_k$ and $b_1 \le \cdots \le b_l$ be sequences of positive integers at most M. Prove that if

$$\sum_{i=1}^{k} a_i^n = \sum_{j=1}^{l} b_j^n \tag{$\star$}$$

for all $1 \le n \le M$, then $k = l$ and $a_i = b_i$ for all $1 \le i \le k$.

Solution

For each $i \in \{1, 2, \ldots, M\}$, let c_i and d_i be the number of "i" in the sequence $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_l)$, respectively. Then the equation $(\star)$ is equivalent to

$$\sum_{i=1}^{M} i^n c_i = \sum_{i=1}^{M} i^n d_i, \quad \forall n \in \{1, 2, \ldots, M\}$$

which can be written as

$$V \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{pmatrix} = V \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ v_M \end{pmatrix}$$

where

$$V = \begin{pmatrix} 1 & 2 & \cdots & M \\ 1^2 & 2^2 & \cdots & M^2 \\ & \vdots & \ddots & \vdots \\ 1^M & 2^M & \cdots & M^M \end{pmatrix}$$

Since V is the Vandermonde matrix whose determinant is nonzero, V is invertible. This shows that $c_i = d_i$ for each $i = 1, 2, \ldots, M$. Hence we conclude that $k = l$ and $a_i = b_i$ for each $i = 1, 2, \ldots, k$. $\square$