A series

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POW2011-1. *Evaluate the sum*

$$\sum_{n=1}^{\infty} \frac{n \sin n}{1+n^2}$$

Solution. First, the series is convergent by the Dirichlet's Test. Note that $\int_0^\infty e^{-nx} \cos x dx = \dfrac{n}{1+n^2}$, and

$$|\sum_{n=1}^{k} e^{-nx} \sin n| \leq \sum_{n=1}^{k} |e^{-nx} \sin n| \leq \sum_{n=1}^{k} |e^{-nx}| \leq \sum_{n=1}^{\infty} e^{-nx} = \frac{1}{e^x - 1}$$

except for $x = 0$, where $\frac{1}{e^x-1}$ is integrable on $[\epsilon, \infty)$ for given $\epsilon > 0$.

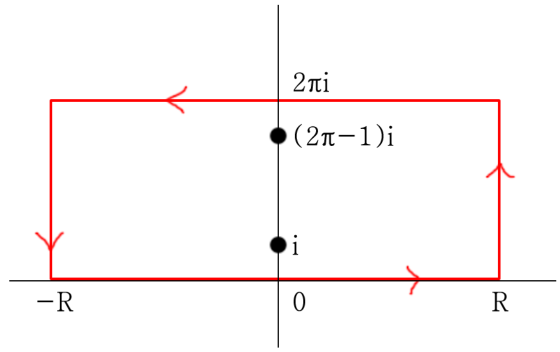
Thus, we can apply the Lebesgue's dominated convergence theorem to obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n \sin n}{1+n^2} &= \sum_{n=1}^{\infty} \sin n \int_0^\infty e^{-nx} \cos x dx \\
&\overset{\text{D.C.T.}}{=} \int_0^\infty \sum_{n=1}^{\infty} e^{-nx} \sin n \cos x dx \\
&= \int_0^\infty \Im\left(\sum_{n=1}^{\infty} e^{-nx} e^{in} \right) \cos x dx \\
&= \int_0^\infty \Im\left(\frac{e^i}{e^x - e^i} \right) \cos x dx \\
&= \int_0^\infty \frac{\sin 1}{2\cosh x - 2\cos 1} \cos x dx \\
&= \frac{\sin 1}{2} \int_0^\infty \frac{\cos x}{\cosh x - \cos 1} dx \qquad (1)
\end{aligned}$$

and let $f(z) \equiv \dfrac{e^{zi}}{\cosh z - \cos 1}$.

$f(z)$ has its singularities at $z = (2\mathbb{Z}\pi \pm 1)i$. Let Γ be a rectangular contour which consists of the intervals $C1 : [-R, R], C2 : [R, R+2\pi i], C3 : [R+2\pi i, -R+2\pi i], C4 : [-R+2\pi i, -R]$ for given real number $R > 0$. Then the followings hold:

Figure 1: A rectangular contour Γ



- $\int_{C_3} f(z)dz = -\int_{C_1} f(z+2\pi i)dz = -\int_{C_1} \frac{e^{(z+2\pi i)i}}{\cosh z - \cos 1} dz = -e^{-2\pi} \int_{C_1} f(z)dz$
- $|\int_{C_2} f(z)dz| \leq 2\pi \max_{z\in C_2} |f(z)| = 2\pi \max_{0\leq t\leq 2\pi} \frac{e^{-t}}{|\cosh(R+ti) - \cos 1|} \to 0$ and $|\int_{C_4} f(z)dz| \leq 2\pi \max_{z\in C_4} |f(z)| = 2\pi \max_{0\leq t\leq 2\pi} \frac{e^{-t}}{|\cosh(-R+ti) - \cos 1|} \to 0$ as $R \to \infty$.

by the estimation lemma because $|\cosh z - \cos 1| \geq |\cosh z| - |\cos 1| \to \infty$ as $\Re(z) \to \infty$ with $0 \leq \Im(z) \leq 2\pi$. Therefore, by the residue theorem,

$$\begin{aligned}\int_\Gamma f(z)dz &= \lim_{R\to\infty} \left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}\right) f(z)dz \\ &= \lim_{R\to\infty} \left(1 - e^{-2\pi}\right) \int_{C_1} f(z)dz = \left(1 - e^{-2\pi}\right) \int_{-\infty}^{\infty} f(z)dz \\ &= 2\pi i \left(Res\left(f(z); i\right) + Res\left(f(z); (2\pi - 1)i\right)\right) \\ &= 2\pi i \left(\frac{e^{-1}}{i \sin 1} - \frac{e^{1-2\pi}}{i \sin 1}\right) = \frac{2\pi(1 - e^{2-2\pi})}{e \sin 1}\end{aligned}$$

so we obtain $\int_{-\infty}^{\infty} f(z)dz = \frac{2\pi(1 - e^{2-2\pi})}{(1 - e^{-2\pi}) e \sin 1}$. Since $\frac{\cos x}{\cosh x - \cos 1}$ is even,

$$\begin{aligned}\int_0^\infty \frac{\cos x}{\cosh x - \cos 1} dx &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos x}{\cosh x - \cos 1} dx \\ &= \frac{1}{2} \Re\left(\int_{-\infty}^{\infty} f(z)dz\right) = \frac{\pi(1 - e^{2-2\pi})}{(1 - e^{-2\pi}) e \sin 1}\end{aligned}$$

and $\sum_{n=1}^{\infty} \frac{n \sin n}{1 + n^2} = \frac{\pi(1 - e^{2-2\pi})}{2(1 - e^{-2\pi}) e} = \frac{\pi \sinh(\pi - 1)}{2 \sinh \pi} \approx 0.570956$ by (1). □