Solution: POW 2010-21

수리과학과 김치헌

Consider $(4n+1) - 3a_{4n+1}$. We have

$$\begin{aligned}(4n+1) - 3a_{4n+1} &= (4n+1) - 3(2n - a_{2n}) \\ &= 3a_{2n} - 2n + 1 \\ &= 3(n - a_n) - 2n + 1 \\ &= n - 3a_n + 1.\end{aligned}$$

Define $m_1 = 1$ and $m_i = 4m_{i-1} + 1$ for $i \geq 2$. Then, $m_1 - 3a_{m_1} = 1$ and so $m_k - 3a_{m_k} = k$. By letting $k = m_{6033}$, $\frac{k}{3} - a_k = 2011 > 2010$.

To prove the second part, I claim that

$$\left|\frac{a_n}{n} - \frac{1}{3}\right| \leq \frac{k}{3 \cdot 2^k}$$

for any $2 \leq 2^k \leq n < 2^{k+1}$. Use induction on n.

Consider the base cases, $n = 2, 3$. Observe that

$$\begin{aligned}\left|\frac{a_2}{2} - \frac{1}{3}\right| &= \frac{1}{6} \leq \frac{1}{3 \cdot 2^1} \\ \left|\frac{a_3}{3} - \frac{1}{3}\right| &= 0 < \frac{1}{3 \cdot 2^1}.\end{aligned}$$

For $n \geq 4$, if $n = 2m$, then

$$\begin{aligned}\left|\frac{a_n}{n} - \frac{1}{3}\right| &= \left|\frac{m - a_m}{2m} - \frac{1}{3}\right| \\ &= \frac{1}{2}\left|\frac{1}{3} - \frac{a_m}{m}\right| \\ &\leq \frac{1}{2}\frac{k-1}{3 \cdot 2^{k-1}} < \frac{k}{3 \cdot 2^k}.\end{aligned}$$

If $n = 2m+1$, then

$$\begin{aligned}
\left|\frac{a_n}{n} - \frac{1}{3}\right| &= \left|\frac{m - a_m}{2m+1} - \frac{1}{3}\right| \\
&= \left|\frac{m+1}{3(2m+1)} - \frac{a_m}{2m+1}\right| \\
&= \left|\frac{1}{3(2m+1)} + \frac{m}{2m+1}\left(\frac{1}{3} - \frac{a_m}{m}\right)\right| \\
&\leq \frac{1}{3 \cdot 2^k} + \frac{1}{2}\left|\frac{a_m}{m} - \frac{1}{3}\right| \\
&\leq \frac{1}{3 \cdot 2^k} + \frac{1}{2}\frac{k-1}{3 \cdot 2^{k-1}} = \frac{k}{3 \cdot 2^k}.
\end{aligned}$$

Since $\frac{k}{2^k}$ goes to 0 as $k \to \infty$,

$$\lim_{n\to\infty} \frac{a_n}{n} = \frac{1}{3}.$$