POW 2010-17 Attempt

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We are to show $\mathrm{tr}((AB)^2) - \mathrm{tr}(A^2B^2) \leq 0$ for two Hermitian matrices A and B. Let A and B be two Hermitian matrices. Then $A^\dagger = A, B^\dagger = B$ by definition, where $A^\dagger$ denotes the complex conjugate of A.

We have

$$\begin{aligned}\mathrm{tr}(X+Y) &= \mathrm{tr}(X) + \mathrm{tr}(Y)\\ \mathrm{tr}(XY) &= \mathrm{tr}(YX)\end{aligned}$$

for any square matrices X, Y.

$$\begin{aligned}\mathrm{tr}(A^2B^2) &= \mathrm{tr}(AABB)\\ &= \mathrm{tr}((AAB)B) = \mathrm{tr}(BAAB) = \mathrm{tr}((BA)(AB))\\ &= \mathrm{tr}(A(ABB)) = \mathrm{tr}(ABBA) = \mathrm{tr}((AB)(BA))\\ \mathrm{tr}((AB)^2) &= \mathrm{tr}(ABAB)\\ &= \mathrm{tr}(A(BAB)) = \mathrm{tr}((BAB)A)\\ &= \mathrm{tr}((BA)^2).\end{aligned}$$

Therefore we have

$$\begin{aligned}\mathrm{tr}((AB)^2) - \mathrm{tr}(A^2B^2) &= \frac{1}{2}(\mathrm{tr}((AB)^2) - \mathrm{tr}((AB)(BA))\\ &\quad - \mathrm{tr}((BA)(AB)) + \mathrm{tr}((BA)^2)\\ &= \frac{1}{2}\,\mathrm{tr}\left((AB)^2 - (AB)(BA) - (BA)(AB) + (BA)^2\right)\\ &= \frac{1}{2}\,\mathrm{tr}((AB - BA)^2).\end{aligned}$$

Now we look at $AB - BA$, where A and B are Hermitian.

$$\begin{aligned}
(AB - BA)^\dagger &= (AB)^\dagger - (BA)^\dagger \\
&= B^\dagger A^\dagger - A^\dagger B^\dagger \\
&= BA - AB \\
&= -(AB - BA).
\end{aligned}$$

Therefore, $AB-BA$ is a skew-Hermitian matrix, eigenvalues of $AB-BA$ are pure imaginary numbers, eigenvalues of $(AB-BA)^2$ are nonpositive real. This gives us

$$\mathrm{tr}((AB - BA)^2) = \sum(\text{eigenvalues of}(AB - BA)^2) \leq 0.$$

which finishes our proof.