Solution: POW 2010-18

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Define $g(x) = e^x f(x)$. Then, $g'(x) = e^x(f(x) + f'(x))$. So it suffices to prove the following:

> Let g be a differentiable function satisfying $\lim_{x\to\infty} \frac{g'(x)}{e^x} = 1$. Then, $\lim_{x\to\infty} \frac{g(x)}{e^x} = 1$.

Remind L'hopital's rule in Calculus I. The problem above has the same form with that, thus proving $g(x)$ goes to infinity as $x \to \infty$ is enough.

We know $\frac{g'(x)}{e^x} \to 1$ as $x \to \infty$. So, there is a number M such that

$$\frac{1}{2} < \frac{g'(x)}{e^x}$$

for any $x > M$. So, for such x,

$$\begin{aligned} g(x) &= \int_M^x g'(t)dt + g(M) \\ &> \int_M^x \frac{e^t}{2} dt + g(M) \\ &= \frac{e^x}{2} + \left(g(M) - \frac{e^M}{2}\right). \end{aligned}$$

Hence $g(x)$ goes to infinity because $\frac{e^x}{2}$ does so.