Monochromatic line

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POW2010-20. *Let X be a finite set of points on the plane such that each point in X is colored with red or blue and there is no line having all points in X. Prove that there is a line L having at least two points of X such that all points in $L \cap X$ have the same color.*

Solution. The following lemma is well-known as a consequence of Euler's formula.

Lemma. *Let G be a connected simple plane graph embedded on the unit sphere, and its edges be colored with red or blue. For a vertex $v \in V(G)$, color change of v indicates a pair of two nearby edges which are adjacent at v, and colors are different. Denote $c(v)$ to be the number of color changes of v. (Figure 1a.) Then, there is a vertex v such that $c(v) \leq 2$.*

(a) $c(v) = 4$ (b) 21 sides (c) At most $2k$ color changes

Proof of Lemma. Suppose not. Note that $c(v)$ is always even. Thus, we may assume that $c(v) \geq 4$ for all $v \in V(G)$. Let f_k be the number of faces of G bounded by k edges. (If its both sides are on the same face, an edge should be counted twice, for G can be not 2-connected. Figure 1b.) Then, we obtain

$$2|E(G)| = \sum_{k=3}^{\infty} k f_k$$

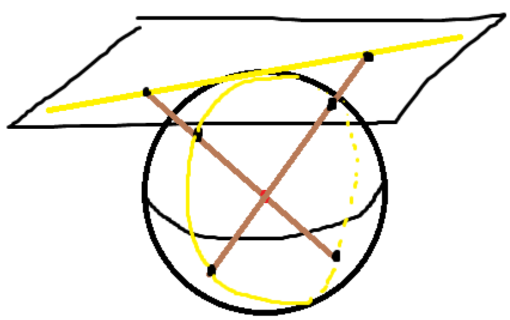
because we count each edge exactly 2 times in the right hand side whenever an edge lies on the boundary or the inside of a face. On the other hand, at most $2k$

color changes can occur at each corner of a face with $2k$ or $2k+1$ sides. (Figure 1c.) Consequently,

$$\begin{aligned}
4V(G) \leq \sum_{v \in V(G)} c(v) &\leq 2f_3 + 4f_4 + 4f_5 + 6f_6 + \cdots \\
&\leq 2f_3 + 4f_4 + 6f_5 + 8f_6 + \cdots \\
&= \sum_{k=3}^{\infty} (2k-4) f_k = 2\sum_{k=3}^{\infty} k f_k - 4 \sum_{k=3}^{\infty} f_k \\
&= 4E(G) - 4F(G)
\end{aligned}$$

holds, so $V(G) - E(G) + F(G) \leq 0$. This is a contradiction to Euler's formula, $V(G) - E(G) + F(G) = 2$. $\square$

Let's project every points in the plane to the unit sphere S^2 in $\mathbb{R}^3$ as the below. Each point of the plane corresponds to a pair of antipodal points in S^2, and each line corresponds to a great circle on S^2. By the condition, not



all points lie on one great circle on S^2. Moreover, we can think of the duality between a pair of antipodal points and a great circle. For each pair of antipodal points v_1, v_2 on S^2, there is the unique great circle orthogonal to $\overline{v_1 v_2}$, vice versa. Thus, the original problem is equivalent to the following statement.

> Let X be a finite set of great circles on S^2 such that each great circle in X is colored with red or blue, and not all great circles in X pass through one point. Then, there is an intersection point which only red (or blue) great circles pass through.

Observe that if two great circles with different colors intersect, the intersection point has at least 4 color changes. Hence, the above statement is proved by the lemma because the graph made by great circles is a simple connected plane graph. By the duality, for any configuration of pairs of antipodal points on S^2, if not all of them lie on one great circle then there is a monochromatic great circle. Therefore, from the projection at the beginning of the proof, for any configuration of points on the plane, if they are not collinear then there is a monochromatic line on the plane, which is the desired result. $\square$