POW 2010-14 Attempt

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We need to show
$$\sum_{k=0}^{n}(-1)^k\binom{2n+2k}{n+k}\binom{n+k}{2k} = (-4)^n \text{ for all } n \in \mathbb{N}$$
which is equivalent to
$$\frac{1}{(-4)^n}\sum_{k=0}^{n}(-1)^k\binom{2n+2k}{n+k}\binom{n+k}{2k} = \sum_{k=0}^{n}\frac{(-1)^k}{(-4)^n}\binom{2n+2k}{n+k}\binom{n+k}{2k} = 1 \text{ for all } n \in \mathbb{N}.$$

Observe $n = 0$, $\sum_{k=0}^{n}\frac{(-1)^k}{(-4)^n}\binom{2n+2k}{n+k}\binom{n+k}{2k} = \frac{(-1)^0}{(-4)^0}\binom{0}{0}\binom{0}{0} = 1$. Therefore it is suffice to show
$$\sum_{k=0}^{n}\frac{(-1)^k}{(-4)^n}\binom{2n+2k}{n+k}\binom{n+k}{2k},$$
given function of n is constant over natural numbers and zero. We are going to denote $\mathbb{N}$ as set of natural numbers and zero for convenience.

Let $F(n,k)$ be defined by
$$F(n,k) := \frac{(-1)^k}{(-4)^n}\binom{2n+2k}{n+k}\binom{n+k}{2k}.$$

Then
$$\begin{aligned}
F(n+1,k) - F(n,k) &= \frac{(-1)^k}{(-4)^{n+1}}\binom{2n+2k+2}{n+k+1}\binom{n+k+1}{2k} - \frac{(-1)^k}{(-4)^n}\binom{2n+2k}{n+k}\binom{n+k}{2k} \\
&= \frac{(-1)^k}{(-4)^n}\binom{2n+2k}{n+k}\binom{n+k}{2k}\left(\frac{(2n+2k+2)(2n+2k+1)}{-4(n+k+1)(n-k+1)} - 1\right) \\
&= F(n,k)\left(-\frac{(2n+2k+1)+(2n-2k+2)}{2(n-k+1)}\right) \\
&= -F(n,k)\frac{4n+3}{2(n-k+1)} \text{ for all } n \text{ and all } k.
\end{aligned}$$

Now, let $G(n,k)$ be defined by
$$G(n,k) := \frac{(4n+3)(2k-1)k}{2(n-k+1)(n+1)(2n+1)}F(n,k), \text{ and see}$$

$$\begin{aligned} G(n,k+1)\Big|_{k=n} &= \frac{(4n+3)(2k+1)(k+1)}{2(n+1)(2n+1)}\frac{(-1)(2n+2k+2)(2n+2k+1)}{(n+k+1)(2k+1)(2k+2)}F(n,k) \\ &= -\frac{(4n+3)(2n+2k+1)}{2(n+1)(2n+1)}F(n,k). \end{aligned}$$

$$\begin{aligned} G(n,k+1)-G(n,k) &= -\frac{(4n+3)(2n+2k+1)}{2(n+1)(2n+1)}F(n,k) \\ &\quad -\frac{(4n+3)(2k-1)k}{2(n-k+1)(n+1)(2n+1)}F(n,k) \\ &= -\frac{4n+3}{2(n+1)(2n+1)}F(n,k)\frac{(2n+2k+1)(n-k+1)+2k^2-k}{n-k+1} \\ &= -\frac{4n+3}{2(n+1)(2n+1)}F(n,k)\frac{2n^2+3n+1}{n-k+1} \\ &= -\frac{4n+3}{2(n-k+1)}F(n,k). \end{aligned}$$

Therefore,

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k) \text{ for all } n\in\mathbb{N}, k\in\mathbb{N}.$$

Sum this equation by all integers k to get

$$\begin{aligned} &\sum_{k=0}^{n}F(n+1,k)-\sum_{k=0}^{n}F(n,k)=\sum_{k=0}^{\infty}F(n+1,k)-\sum_{k=0}^{\infty}F(n,k) \\ =\ &\sum_{k=0}^{\infty}(G(n,k+1)-G(n,k))=\lim_{k\to\infty}G(n,k)-G(n,0) \\ =\ &\lim_{k\to\infty}G(n,k)=0,\ \text{since } F(n,k)=0 \text{ for } k>n. \end{aligned}$$

$$\therefore \sum_{k=0}^{n}F(n+1,k)=\sum_{k=0}^{n}F(n,k) \text{ for all } n.$$

Therefore, $\sum_{k=0}^{n}F(n,k)$ is constant over n, resulting

$$\sum_{k=0}^{n}(-1)^k\binom{2n+2k}{n+k}\binom{n+k}{2k}=(-4)^n \text{ for all } n\in\mathbb{N}.$$

Note : This proof is guided by the book 'A=B' by Marko Petkovsek, Herbert Wilf and Doron Zeilberger, and $G(n,k)$ is derived using MAPLE.