POW 2010-12 Attempt

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Let A be a $n \times n$ matrix and J be a $n \times n$ diagonal matrix. Denote diagonal entries of J by $x_1, x_2, \ldots, x_n$ such that $x_i \in \{-1, 1\}$ for $i = 1, 2, \ldots, n$.

Our goal is to show constructing J by determining x_is such that $\det(A + J)$ is nonzero is possible. Then there exist an invertible matrix $A + J$. We show such a construction is possible by actually constructing such J by induction.

Define B_k by upper-left submatrix of $A + J$ of size $k \times k$. i.e., (i, j)th entry of B_k $(i, j = 1, 2, \ldots, k)$ is equal to (i, j)th entry of $A + J$. We can observe B_1 is 1×1 matrix and $B_n = A + J$.

First we observe a base case. $B_1 = (a_{11} + x_1)$ where $A = \{a_{ij}\}$. By Fundamental Theorem of Algebra, $a_{11} + x_1 = 0$, a polynomial in x_1 of degree 1, has at most one solution. Thus, at least one of $x_1 \in \{-1, 1\}$ makes $a_{11} + x_1$ nonzero, so that $\det B_1 \neq 0$. Fix x_1 so that $\det B_1 \neq 0$.

Assume $\det B_{i-1} \neq 0$. Then

$$\det B_i = (a_{ii} + x_i) \det B_{i-1} + C$$

where C is constant respect to x_i. Again, righthand side of above equation is a polynomial in x_i of degree 1. By FTA, this polynomial has at most one solution, thus we can choose $x_i = 1$ or $x_i = -1$ so that $\det B_i \neq 0$. Fix(substitute) x_i so that $\det B_i \neq 0$.

By induction step, we could construct B_n such that $\det B_n = \det(A + J) \neq 0$ so that $A + J$ is invertible and each diagonal entry of J is 1 or -1.