PROVING A BINOMIAL IDENTITY USING THE CAUCHY THEOREM

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The problem is to prove the identity

$$\sum_{k=0}^{n}(-1)^k \binom{2n+2k}{n+k}\binom{n+k}{2k} = (-4)^n \tag{0.1}$$

for all positive integers n. Our plan of attack is to use the Cauchy integral formula.

Let us begin by observing that by a simple bijection argument, we can rewrite the product of binomial coefficients as :

$$\binom{2n+2k}{n+k}\binom{n+k}{2k} = \binom{2n+2k}{2k}\binom{2n}{k} \tag{0.2}$$

Indeed, both sides could be interpreted as choosing out of $2n+2k$ balls $2k$ balls to be put in one box and $n+k-2k = n-k$ balls to be put in another box.

Using this, and multiplying both sides by $(-1)^n$, it suffices to show

$$\sum_{k=0}^{n}(-1)^{n-k} \binom{2n+2k}{2k}\binom{2n}{n-k} = 4^n. \tag{0.3}$$

Note that $\binom{2n+2k}{2k}$ is the coefficient in front of z^{2k} in $(1+z)^{2n+2k}$. By the Cauchy integral formula, we have

$$\binom{2n+2k}{2k} = \frac{1}{2\pi i}\oint_\Gamma \frac{(1+z)^{2n+2k}}{z^{2k+1}}\mathrm{d}z \tag{0.4}$$

where Γ is a small closed contour surrounding 0.

Using (0.2), (0.4), and we can rewrite the LHS of (0.3) as

$$\frac{1}{2\pi i}\oint_\Gamma (-1)^{n-k}\frac{(1+z)^{2n+2k}}{z^{2k+1}}\binom{2n}{n-k}\mathrm{d}z. \tag{0.5}$$

Replacing the index $n-k$ by k, we get

$$\begin{aligned}
(0.5) &= \frac{1}{2\pi i}\sum_{k=0}^{n}\oint_\Gamma (-1)^k \frac{(1+z)^{4n-2k}}{z^{2n-2k+1}}\binom{2n}{k}\mathrm{d}z \\
&= \frac{1}{2\pi i}\oint_\Gamma \frac{(1+z)^{4n}}{z^{2n+1}}\sum_{k=0}^{n}(-1)^k\binom{2n}{k}\left(\frac{1+z}{z}\right)^{-2k}\mathrm{d}z. \\
&= \frac{1}{2\pi i}\oint_\Gamma \frac{(1+z)^{4n}}{z^{2n+1}}\sum_{k=0}^{n}(-1)^k\binom{2n}{k}\left(\frac{z}{1+z}\right)^{2k}\mathrm{d}z.
\end{aligned}$$

Now let us look at the sum on the last line. If the sum were over indices from $k=0$ to $k=2n$, then it will be equal to $(1-(z/1+z)^2)^{2n}$ by the binomial theorem; however, the problem here is that the sum is only up to $k=n$. What saves us here is the fact that the terms corresponding to $k>n$ *does not contribute* to the Cauchy integral. To see this, note that we are dividing by z^{2n+1}, and in a term with $k > n$ we have z^{2k} in the numerator, and $2k > 2n+1$. The factor involving powers of $1+z$ does not have a residue at 0, so as a consequence the term with $k > n$ has no residue at 0. (In fact, it is holomorphic on the whole complex plane) Therefore, we can harmlessly add the remaining terms from $k = n+1$ to $k = 2n$, and we are led to

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\begin{aligned}
(0.5) &= \frac{1}{2\pi i} \oint_\Gamma \frac{(1+z)^{4n}}{z^{2n+1}} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \left(\frac{z}{1+z} \right)^{2k} \mathrm{d}z \\
&= \frac{1}{2\pi i} \oint_\Gamma \frac{(1+z)^{4n}}{z^{2n+1}} \left(1 - \left(\frac{z}{1+z} \right)^2 \right)^{2n} \mathrm{d}z \\
&= \frac{1}{2\pi i} \oint_\Gamma \frac{(1+z)^{4n}}{z^{2n+1}} \left(\frac{1+2z}{(1+z)^2} \right)^{2n} \mathrm{d}z \\
&= \frac{1}{2\pi i} \oint_\Gamma \frac{(1+2z)^{2n}}{z^{2n+1}} \mathrm{d}z \\
&= 2^{2n} = 4^n,
\end{aligned}
$$

where the last line is again by the Cauchy integral formula. This proves (0.3).