Combinatorial Identity

Minjae Park

POW2010-14. *Let n be a positive integer. Prove that*

$$\sum_{k=0}^{n}(-1)^k\binom{2n+2k}{n+k}\binom{n+k}{2k}=(-4)^n$$

First Solution *(Combinatorial Method).* I will count the number of ways to assign 0 or 1 to $2n$ ordered cards. First, it's trivial that this number is exactly same as the number of $2n$-digits binary numbers, which is $2^{2n}=4^n$.

Now, let's count this number in a different way. Consider the situation that $2n$ ordered "dummy" cards are inserted to the original $2n$ cards. Then we have $4n$ cards overall, and I will choose $2n$ cards among them. The number of ways to choose $2n$ cards is $\binom{4n}{2n}$ and I will assign "1" to the selected cards, and "0" to the others. After this selection, if I discard the dummies, then $2n$-digits binary number is obtained. To visualize this situation, I will use the following notation

$$a_1a_2\cdots a_{2n}|b_1b_2\cdots b_{2k}$$

where a_i, b_j are 0 or 1 for all i and j, to depict the situation when i-th original card is assigned to a_i (1 if selected, and 0 if not), and j-th dummy is assigned to b_j (1 if selected, and 0 if not).

It can be easily observed that some selections result the same output (as a $2n$-digits binary number.) In particular, this overlap happens when I choose the same original cards, but different dummies. For example, the two situations

$$010011|000111 \qquad 010011|111000$$

will result the same output, "010011". Moreover, there are exactly $\binom{6}{3}$ situations resulting "010011" which is the number of ways to choose three "1" among six dummies. Thus, I will subtract this number of situations overlapped to other one. To achieve this, I will discard all situations which the binary number of "dummy part" has "10" because the only situation that does not contain "10" in its dummy part has a form only like this.

$$a_1a_2\cdots a_{2n}|000\cdots111$$

To count the situations having "10" in their dummy parts, let's consider the situation

$$a_1a_2\cdots a_{2n}|b_1b_2\cdots b_{2n-2}$$

and put “10” to each $2n-1$ locations before and after each digits of the dummy part. For example, the situation 0111|01 will yields the 3 situations having “10” in their dummy parts.

0111|**10**01 0111|0**10**1 0111|01**10**

Here, I only assign $2n-1$ “1”s among $4n-2$ cards, since I will use one “1” to put “10”. This number can be calculated as $\binom{4n-2}{2n-1}{}_{2n-1}H_1$ (where H stands for combination with repetition.) However, if the dummy part of $a_1a_2\cdots a_{2n}|b_1b_2\cdots b_{2n-2}$ already contains “10”, it is overlapped by putting “10” in another location. For instance, 0111|10 yields three situations which dummy parts have “10”, but two of them are exactly same as 0111|1010.

Hence, I need to add the number of situations which dummy parts having two “10”s. Similarly, think of the situation

$$a_1a_2\cdots a_{2n}|b_1b_2\cdots b_{2n-4}$$

and I will put two “10”s to each $2n-3$ locations before and after each digits of the dummy part. Note that two “10”s can be inserted to the exactly same location. This is why I used combination with repetition in the previous case. In this situations, I only assign $2n-2$ “1”s among $4n-4$ cards since two “1”s will be used when I insert two “10”s to the dummy part. This number can be calculated as $\binom{4n-4}{2n-2}{}_{2n-3}H_2$. However, it also over-counts some situations which dummy parts have three “10”s.

To obtain the actual number, we can continue these alternative adding and subtracting. Remark that the dummy part of one situation can contain at most n “10”s. Consequently, continuing this argument, we conclude that the whole number of ways to assign 0 or 1 to the original $2n$ cards is

$$\sum_{l=0}^{n}(-1)^l\binom{4n-2l}{2n-l}{}_{2n-2l+1}H_l$$

By substituting $k=n-l$ and using the fact that ${}_nH_r=\binom{n+r-1}{r}$, the number is same as

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^{n-k}\binom{2n+2k}{n+k}{}_{2k+1}H_{n-k}\\ =&\sum_{k=0}^{n}(-1)^{n-k}\binom{2n+2k}{n+k}\binom{n+k}{n-k}=\sum_{k=0}^{n}(-1)^{n-k}\binom{2n+2k}{n+k}\binom{n+k}{2k}\end{aligned}$$

At the beginning we counted this number as 4^n, so the following equation holds.

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{2n+2k}{n+k}\binom{n+k}{2k}=4^n$$

Therefore, the desired identity is immediately derived. □

Second Solution *(Using Generating Function).* Let's denote $[t^n]f(t) \equiv f_n$ where $f(t)$ be the generating function of the sequence $\{f_i\}_{i\in\mathbb{N}}$.

Lemma 1. *Let $f(t)$ be the generating function of the sequence $\{f_i\}_{i\in\mathbb{N}}$. Then,*

$$\sum_{k=0}^{\infty}\binom{n+k}{2k}f_k = [t^n]\frac{1}{1-t}f(\frac{t}{(1-t)^2}).$$

Proof. First, from the basic combinatorial identities,

$$\begin{aligned}\binom{n+k}{2k} = \binom{n+k}{n-k} &= (-1)^{-k}\binom{-2k-1}{n-k} \\ &= [t^{n-k}]\frac{1}{(1-t)^{2k+1}} \\ &= [t^n]\frac{1}{1-t}\left(\frac{t}{(1-t)^2}\right)^k \qquad (1)\end{aligned}$$

holds. By (1), we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\binom{n+k}{2k}f_k &= \sum_{k=0}^{\infty}[t^n]\frac{1}{1-t}\left(\frac{t}{(1-t)^2}\right)^k[s^k]f(s) \\ &= [t^n]\frac{1}{1-t}\sum_{k=0}^{\infty}[s^k]f(s)\left(\frac{t}{(1-t)^2}\right)^k \\ &= [t^n]\frac{1}{1-t}f(\frac{t}{(1-t)^2}),\end{aligned}$$

which is the desired result. □

Let's denote an operator $\mathcal{G}$, formally well-defined and satisfying $\mathcal{G}([t^n]f(t)) = f(t)$ where $f(t)$ be the generating function of the sequence $\{f_i\}_{i\in\mathbb{N}}$. Then, the following lemma is one result of Lagrange's Inversion Theorem.

Lemma 2 (Diagonalization Rule of Lagrange's Inversion Theorem)**.** *Let F(t) be any formal power series, then*

$$\mathcal{G}([t^n]F(t)\phi(t)^n) = \left[\frac{F(w)}{1-t\phi'(w)}\middle| w = t\phi(w)\right].$$

I will omit the proof. The notation $[f(w)|w = g(t)]$ is a linearization of $f(w)|_{w=g(t)}$ and denotes the substitution of g(t) to every occurrence of w in $f(w)$ (that is, $f(g(t))$). In particular, $w = t\phi(w)$ is to be solved in $w = w(t)$ and w has to be substituted in the expression on the left of the | sign (These notations used in Lemma 2 are borrowed from the reference).

By the Lemma 2, the following identity can be derived.

$$\begin{aligned}
f_k \equiv (-1)^k \binom{2n+2k}{n+k} &= (-1)^n [t^k] \frac{1}{t^n}(1-t)^{2n+2k} \\
&= (-1)^n [t^k] \frac{(1-t)^{2n}}{t^n}((1-t)^2)^k \\
&= (-1)^n [t^k] \left[\frac{(1-w)^{2n}}{w^n} \frac{1}{1+2t(1-w)} \middle| w = t(1-w)^2 \right] \\
&= (-1)^n [t^k] \left[\frac{(1-w)^{2n}}{w^n} \frac{1-w}{1-w+2w} \middle| w = t(1-w)^2 \right] \\
&= (-1)^n [t^k] \left[\frac{(1-w)^{2n+1}}{w^n(1+w)} \middle| w = t(1-w)^2 \right]
\end{aligned}$$

Thus, applying Lemma 1 results

$$\begin{aligned}
&\sum_{k=0}^{\infty} (-1)^k \binom{2n+2k}{n+k}\binom{n+k}{2k} = \sum_{k=0}^{\infty} \binom{n+k}{2k} f_k \\
=&(-1)^n [t^n] \frac{1}{1-t} \left[\frac{(1-w)^{2n+1}}{w^n(1+w)} \middle| w = \frac{t}{(1-t)^2}(1-w)^2 \right].
\end{aligned} \tag{2}$$

Note that $w = t$ is a solution of $w = \frac{t}{(1-t)^2}(1-w)^2$. By replacing $w = t$ to (2),

$$\begin{aligned}
&\sum_{k=0}^{\infty} (-1)^k \binom{2n+2k}{n+k}\binom{n+k}{2k} = \sum_{k=0}^{\infty} \binom{n+k}{2k} f_k \\
=&(-1)^n [t^n] \frac{1}{1-t} \frac{(1-t)^{2n+1}}{t^n(1+t)} = (-1)^n [t^{2n}] \frac{(1-t)^{2n}}{1+t} \\
=&(-1)^n [t^{2n}] \left(\binom{2n}{0} - \binom{2n}{1} t + \binom{2n}{2} t^2 - \cdots \right) (1 - t + t^2 - \cdots) \\
=&(-1)^n \left(\binom{2n}{0} + \binom{2n}{1} + \binom{2n}{2} + \cdots + \binom{2n}{2n} \right) \\
=&(-1)^n 2^{2n} = (-4)^n
\end{aligned}$$

is obtained. Therefore, since $\binom{n+k}{2k} = 0$ if $k > n$,

$$\sum_{k=0}^{n} (-1)^k \binom{2n+2k}{n+k}\binom{n+k}{2k} = \sum_{k=0}^{\infty} (-1)^k \binom{2n+2k}{n+k}\binom{n+k}{2k} = (-4)^n. \quad \square$$

Reference

- Lagrange Inversion: When and How (D. Merlini, et al.)