Upper bound

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POW2010-13. *Prove that there is a constant C such that*

$$\sup_{A<B} \int_A^B \sin(x^2+yx)dx \le C$$

for all y.

Note that

$$\begin{aligned}\sup_{\substack{A<B\\ y\in\mathbb{R}}} \int_A^B \sin(x^2+yx)dx &= \sup_{\substack{A<B\\ y\in\mathbb{R}}} \int_A^B \sin\left((x+\frac{y}{2})^2 - \frac{y^2}{4}\right) dx \\ &= \sup_{\substack{A<B\\ z\in\mathbb{R}}} \int_A^B \sin(x^2-z^2)dx\end{aligned}$$

because a translation parallel to the x-axis does not affect the supremum of the integration.

First Solution. For a fixed number $z \in \mathbb{R}$, let $f(x) \equiv \sin(x^2-z^2)$ where $x \ge 0$. Since $\sin(x^2-z^2)$ is an even function for x, this proof will show that there is a constant C' such that $\sup_{\substack{0\le A<B\\ z\in\mathbb{R}}} \int_A^B f(x)dx \le C'$, so if we set $C = 2C'$, then

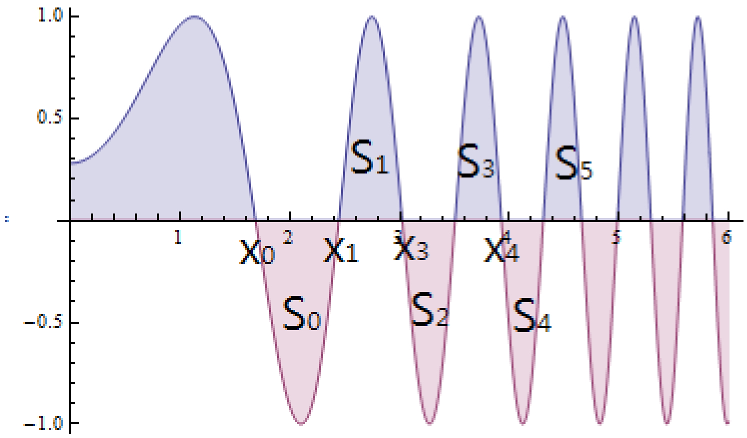
$$\sup_{\substack{A<B\\ z\in\mathbb{R}}} \int_A^B \sin(x^2-z^2)dx \le 2 \sup_{\substack{0\le A<B\\ z\in\mathbb{R}}} \int_A^B f(x)dx \le 2C' = C$$

holds.

Let n be the integer satisfying $n\pi \le z^2 < (n+1)\pi$. Note that $f(x)$ vanishes if and only if $x^2 - z^2 = k\pi$ for some $k \in \mathbb{Z}$. Hence, if we denote x_k be the $(k+1)$th root of f for $k = 0, 1, \cdots$, then $x_k^2 = z^2 + (k-n)\pi$, and $x_0 = \sqrt{z^2 - n\pi}$. Now we can prove the following lemma.

Lemma. *Let $S_k = \int_{x_k}^{x_{k+1}} f(x)dx$. Then $|S_{k+1}| \le |S_k|$ for all non-negative integer k.*

Figure 1: A figure showing decreasing sequence of area $|S_k|$.



Proof of Lemma. By substituting the integration with $u = x^2 - z^2$, and applying the integration by parts, we obtain

$$\begin{aligned}
S_k &= \int_{x_k}^{x_{k+1}} f(x)dx = \int_{x_k}^{x_{k+1}} \sin(x^2 - z^2)dx \\
&= \frac{1}{2}\int_{(k-n)\pi}^{(k-n+1)\pi} \frac{\sin u}{\sqrt{z^2+u}}du \\
&= \frac{1}{2}\left(\left[\frac{-\cos u}{\sqrt{z^2+u}}\right]_{(k-n)\pi}^{(k-n+1)\pi} - \frac{1}{2}\int_{(k-n)\pi}^{(k-n+1)\pi} (z^2+u)^{-\frac{3}{2}}\cos u\, du\right) \\
&= \frac{1}{2}\left(\frac{\cos(k-n)\pi}{x_k} - \frac{\cos(k-n+1)\pi}{x_{k+1}} - \frac{1}{2}\cos\bar{u}\int_{(k-n)\pi}^{(k-n+1)\pi} (z^2+u)^{-\frac{3}{2}}du\right) \\
&= \frac{1}{2}\left(\frac{\cos(k-n)\pi}{x_k} - \frac{\cos(k-n+1)\pi}{x_{k+1}} + \cos\bar{u}\left[\frac{1}{z^2+u}\right]_{(k-n)\pi}^{(k-n+1)\pi}\right) \\
&= \frac{1}{2}\left(\frac{\cos(k-n)\pi}{x_k} - \frac{\cos(k-n+1)\pi}{x_{k+1}} + \cos\bar{u}\left(\frac{1}{x_{k+1}} - \frac{1}{x_k}\right)\right)
\end{aligned}$$

The last equality holds for some $\bar{u} \in ((k-n)\pi, (k-n+1)\pi)$ by the mean value theorem for integration. Thus, by the triangle inequality,

$$|S_k| \le \frac{1}{2}\left(\frac{1}{x_k} + \frac{1}{x_{k+1}} + \frac{1}{x_k} - \frac{1}{x_{k+1}}\right) = \frac{1}{x_k}$$

holds since $|\cos x| \le 1$. For a lower bound for $|S_k|$, we need to consider two cases.

[Case 1] $k-n$ **is even.**

In this case, $f(x)$ is locally increasing near the point x_k and decreasing near the point x_{k+1}. Consequently, $S_k > 0$, $\cos(k-n)\pi = 1$, $\cos(k-n+1)\pi = -1$, and

$$\begin{aligned} S_k &= \frac{1}{2}\left(\frac{1}{x_k} + \frac{1}{x_{k+1}} + \cos\bar{u}\left(\frac{1}{x_{k+1}} - \frac{1}{x_k}\right)\right) \\ &\geq \frac{1}{2}\left(\frac{1}{x_k} + \frac{1}{x_{k+1}} + \left(\frac{1}{x_{k+1}} - \frac{1}{x_k}\right)\right) \\ &= \frac{1}{x_{k+1}} > 0 \end{aligned}$$

which implies $|S_k| \geq \frac{1}{x_{k+1}}$.

[Case 2] $k - n$ **is odd.**

In this case, $f(x)$ is locally decreasing near the point x_k and increasing near the point x_{k+1}. Consequently, $S_k < 0$, $\cos(k-n)\pi = -1$, $\cos(k-n+1)\pi = 1$, and

$$\begin{aligned} -S_k &= \frac{1}{2}\left(\frac{1}{x_k} + \frac{1}{x_{k+1}} - \cos\bar{u}\left(\frac{1}{x_{k+1}} - \frac{1}{x_k}\right)\right) \\ &\geq \frac{1}{2}\left(\frac{1}{x_k} + \frac{1}{x_{k+1}} + \left(\frac{1}{x_{k+1}} - \frac{1}{x_k}\right)\right) \\ &= \frac{1}{x_{k+1}} > 0 \end{aligned}$$

which implies $|S_k| \geq \frac{1}{x_{k+1}}$.

Therefore,

$$|S_{k+1}| \leq \frac{1}{x_{k+1}} \leq |S_k|$$

holds for all non-negative integer k. □

Corollary. *For a given number* $A > x_0$, $\int_A^B f(x)dx \leq |S_0|$ *for all* $B > A$.

Proof of Corollary. Let k be the smallest positive integer such that $x_k \geq A$ and $S_k < 0$. From the lemma,

$$\int_{x_k}^{B} f(x)dx \leq 0$$

for all $B > A$. Thus,

$$\int_A^B f(x)dx = \int_A^{x_k} f(x)dx + \int_{x_k}^{B} f(x)dx \leq S_{k-1} \leq |S_0|$$

holds by applying the lemma again. □

By the corollary, we obtain

$$\sup_{0\le A<B}\int_A^B f(x)dx \le \left|\int_0^{x_0} f(x)dx\right| + |S_0| = \int_0^{x_1} |f(x)|\, dx$$

for a given number $z \in \mathbb{R}$ because f does not change its sign on the interval neither $(0, x_0)$ nor (x_0, x_1). Remark that $x_1 = \sqrt{z^2 + (1-n)\pi}$, so $\pi \le x_1^2 \le 2\pi$ by the choosing of n. Since $|f(x)| \le 1$ for all $x \ge 0$,

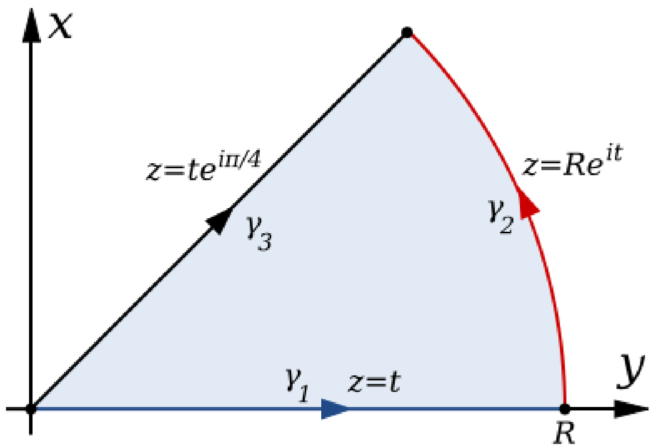
$$\sup_{0\le A<B}\int_A^B f(x)dx \le \int_0^{x_1} |f(x)|\, dx \le \sqrt{2\pi}$$

holds. The upper bound of the supremum $\sqrt{2\pi}$ does not depend on the value of z, so $\sup_{\substack{0\le A<B\\ z\in\mathbb{R}}}\int_A^B f(x)dx \le \sqrt{2\pi}$. Therefore, for a constant $C = 2\sqrt{2\pi}$,

$$\begin{aligned}\sup_{\substack{A<B\\ y\in\mathbb{R}}}\int_A^B \sin(x^2+yx)dx &= \sup_{\substack{A<B\\ z\in\mathbb{R}}}\int_A^B \sin(x^2-z^2)dx \\ &\le 2\sup_{\substack{0\le A<B\\ z\in\mathbb{R}}}\int_A^B f(x)dx \le 2\sqrt{2\pi} = C\end{aligned}$$

holds, and this proves the original proposition. □

Figure 2: This figure is from a Wikipedia entry.



Second Solution. For a fixed $z \in \mathbb{R}$, let $f(x) = e^{(x^2-z^2)i} = e^{x^2 i}e^{-z^2 i}$ be a complex-valued function for $x \in \mathbb{R}$. Consider the contour Γ_R in the [Figure 2] for some $R \leq 0$. Since $f(x)$ has no singularity, $\int_{\Gamma_R} f(x)dx = 0$ by the residue theorem. Thus,

$$\begin{aligned} 0 = \int_{\Gamma_R} f(x)dx &= \left(\int_{\gamma_1} + \int_{\gamma_2} - \int_{\gamma_3}\right) f(x)dx \\ &= e^{-z^2 i}\left(\int_0^R e^{it^2}dt + \int_0^{\frac{\pi}{4}} \exp(iR^2e^{2it})iRe^{it}dt - \int_0^R e^{-t^2}e^{\frac{\pi}{4}i}dt\right) \end{aligned}$$

holds, and

$$e^{-z^2 i}\int_0^R e^{it^2}dt = e^{-z^2 i}\left(e^{\frac{\pi}{4}i}\int_0^R e^{-t^2}dt - \int_0^{\frac{\pi}{4}} \exp(iR^2e^{2it})iRe^{it}dt\right). \qquad (1)$$

Note that $\left|\int_a^b G(t)dt\right| \leq \int_a^b |G(t)|\,dt$ for any continuous complex-valued function G(t), and $\sin(2t) \geq t$ for all $t \in [0, \frac{\pi}{4}]$. Hence, if we let $G(t) = \exp(iR^2e^{2it})iRe^{it}$, then

$$\begin{aligned} \left|\int_0^{\frac{\pi}{4}} G(t)dt\right| &\leq \int_0^{\frac{\pi}{4}} |G(t)|\,dt \\ &= \int_0^{\frac{\pi}{4}} Re^{-R^2\sin(2t)}dt \\ &\leq \int_0^{\frac{\pi}{4}} Re^{-R^2 t}dt = \frac{1 - e^{-\frac{R^2\pi}{4}}}{R} \end{aligned}$$

If $R \geq 1$, then $\frac{1-e^{-\frac{R^2\pi}{4}}}{R} \leq 1$. If $R < 1$, then $\frac{R^2\pi}{4} < 1$, so $e^{-\frac{R^2\pi}{4}} > 1 - \frac{R^2\pi}{4}$ holds by the Taylor expansion. Thus, it's easy to check that

$$\left| \int_0^{\frac{\pi}{4}} G(t)dt \right| \leq \frac{1 - e^{-\frac{R^2\pi}{4}}}{R} \leq 1 \tag{2}$$

for all $R \geq 0$.

Next, by comparing the imaginary parts of (1), we obtain

$$\begin{aligned}
\int_0^R \sin(t^2 - z^2)dt &= \Im\left(\int_0^R e^{(t^2-z^2)i}dt\right) \\
&= \Im\left(\int_0^R e^{(\frac{\pi}{4}-z^2)i} \int_0^R e^{-t^2}dt\right) - \Im\left(e^{-z^2 i}\int_0^{\frac{\pi}{4}} G(t)dt\right) \\
&= \sin(\frac{\pi}{4} - z^2)\int_0^R e^{-t^2}dt - \Im\left(e^{-z^2 i}\int_0^{\frac{\pi}{4}} G(t)dt\right).
\end{aligned}$$

It is very well-known as the Gaussian integral that $\int_0^R e^{-t^2}dt \leq \int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$ for all $R \geq 0$. Consequently, by applying the triangle inequality and (2),

$$\begin{aligned}
\left|\int_0^R \sin(t^2 - z^2)dt\right| &\leq \left|\sin(\frac{\pi}{4} - z^2)\int_0^R e^{-t^2}dt\right| + \left|\Im\left(e^{-z^2 i}\int_0^{\frac{\pi}{4}} G(t)dt\right)\right| \\
&\leq \left|\int_0^R e^{-t^2}dt\right| + \left|e^{-z^2 i}\int_0^{\frac{\pi}{4}} G(t)dt\right| \\
&\leq \frac{\sqrt{\pi}}{2} + 1
\end{aligned}$$

and the following inequality is immediately derived,

$$\begin{aligned}
\left|\int_A^B \sin(t^2 - z^2)dt\right| &= \left|\int_0^B \sin(t^2 - z^2)dt - \int_0^A \sin(t^2 - z^2)dt\right| \\
&\leq \left|\int_0^B \sin(t^2 - z^2)dt\right| + \left|\int_0^A \sin(t^2 - z^2)dt\right| \\
&\leq \sqrt{\pi} + 2
\end{aligned}$$

for any $A, B \geq 0$. Since the number $\sqrt{\pi} + 2$ is unrelated with the value of z, we have

$$\sup_{\substack{0 \leq A < B \\ z \in \mathbb{R}}} \int_A^B \sin(x^2 - z^2)dx \leq \sqrt{\pi} + 2$$

Therefore, if we choose a constant $C = 2\sqrt{\pi} + 4$,

$$\begin{aligned}
\sup_{\substack{A<B \\ y\in\mathbb{R}}} \int_A^B \sin(x^2 + yx)dx &= \sup_{\substack{A<B \\ z\in\mathbb{R}}} \int_A^B \sin(x^2 - z^2)dx \\
&\leq 2 \sup_{\substack{0\leq A<B \\ z\in\mathbb{R}}} \int_A^B \sin(x^2 - z^2)dx \\
&\leq 2(\sqrt{\pi} + 2) = C
\end{aligned}$$

which is the desired result. $\square$