POW 2010-9 No zeros far away

정성구

Given $M>0$ choose N s.t if $n>N$,

$\sum_{k=n+1}^{\infty} \frac{1}{k!M^k} < e^{-\frac{1}{M}}$

(since $\sum_{k=0}^{\infty} \frac{1}{k!M^k} = e^{\frac{1}{M}}$, $\sum_{k=n+1}^{\infty} \frac{1}{k!M^k} \to 0$ as $n \to \infty$)

If $n>N$, $|z| \geq M$,

$$
\begin{aligned}
|f_n(z)| &= \left| e^{\frac{1}{z}} - \sum_{k=n+1}^{\infty} \frac{1}{k!z^k} \right| \\
&\geq \left| e^{\frac{1}{z}} \right| - \left| \sum_{k=n+1}^{\infty} \frac{1}{k!z^k} \right| \\
&\geq e^{-\frac{1}{M}} - \sum_{k=n+1}^{\infty} \frac{1}{k!|z|^k} \qquad \left(\because \left|e^{\frac{1}{z}}\right| = e^{Re(\frac{1}{z})} \geq e^{-\frac{1}{M}} \text{ since } \left|\frac{1}{z}\right| \leq \frac{1}{M}\right) \\
&\geq e^{-\frac{1}{M}} - \sum_{k=n+1}^{\infty} \frac{1}{k!M^k} > 0
\end{aligned}
$$

$\therefore$ all roots of $f_n(z)$ are in $|z|<M$ if $n>N$