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2009학번 서기원

LEMMA. A matrix is nilpotent if and only if its characteristic polynomial is x^n.

proof) ($\Rightarrow$) Let A be a nilpotent matrix. Assume $A^k=0$. Let λ be an eigenvalue of A. Then $Ax=\lambda x$ for some non-zero eigenvector x. Then,

$A^k x = A^{k-1}\times Ax = A^{k-1}\times \lambda x = \lambda A^{k-1}x = \ldots = \lambda^{k-1}Ax = \lambda^k x$

Since $A^k=0$ and x is non-zero vector, $\lambda^k=0$. Thus, $\lambda=0$.

Since we chose an arbitrary eigenvalue, all eigenvalues are equal to zero.

Thus, its characteristic polynomial is $(x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_n)=(x-0)^n=x^n$

($\Leftarrow$) Suppose that all eigenvalues of matrix A are zero. Then the characteristic polynomial of A: $\det(\lambda I - A)=\lambda^n=0$. It now follows from the Cayley-Hamilton theorem that $A^n=0$. Thus, A is a nilpotent matrix.

Let $X=\left\{A\in M_{n\times n} : A^m=0 \text{ for } some\ positive\ integer\ \ m\right\}$.

Assume that matrix A is in the closure of X, that is, for every ϵ, there exist some element of X, let M, such that $d(A,M)<\epsilon$.

Now, consider the characteristic polynomial of A, $p(A)$. Let $p(x)=x^n+\sum_{i=0}^{n-1}c_i x^i$.

Assume that $c_k=c\neq 0$ for some integer $k\in\{0,1,\ldots,n-1\}$. We know that every coefficient of characteristic polynomial can be represented as a continuous function of a_{ij} $(i,j=1,2,\ldots,n)$. Let c_k=$f(a_{11},a_{12},\ldots,a_{nn})$ (f represents a value of k-th coefficient of characteristic polynomial). Since f is continuous,

there exist ϵ such that if $|x_{ij}-a_{ij}|<\epsilon$ for all i,j, then

$|f(x_{11},x_{12},\cdots,x_{nn})-f(a_{11},a_{12},\cdots a_{nn})|=|f(x_{11},x_{12},\cdots,x_{nn})-c|<|c| \Rightarrow f(x_{11},x_{12},\cdots,x_{nn})\neq 0$ $\cdots(1)$

Now, by assumption, there exist matrix $M=(m_{ij})\in X$ such that $\sum_{i,j}|a_{ij}-m_{ij}|=d(A,M)<\epsilon$. Then, by (1), $f(m_{11},m_{12},\cdots,m_{nn})\neq 0$. It means that k-th coefficient of characteristic polynomial of M is not 0. But, it is contradiction since characteristic polynomial of M, nilpotent matrix, is x^n by LEMMA. Thus, assumption is wrong, so $c_k=0$ for all $k\in\{0,1,\ldots,n-1\}$. Therefore, characteristic polynomial of A is just x^n.

Then, by LEMMA, we can obtain that A is nilpotent matrix. Thus $A\in X$. Since every element in the closure of X is an element of X, we can conclude that X is a closed set.

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