POW 2010-7 Cardinality

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For convenience, we use x for x^{-1} (cardinality is unchanged)

We will only consider $x \in (\frac{1}{2}, 1)$

Given $x \in (\frac{1}{2}, 1)$, constructing $\{a_n\}$ s.t $\sum_{n=1}^{\infty} a_n x^n = 1$ must be the following way:

$$a_n = \begin{cases} 1 & \text{if } x^{n+1} + x^{n+2} + \cdots < 1 - (a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}) \cdots (a) \\ 0 & \text{if } x^n > 1 - (a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}) \cdots (b) \\ 0 \text{ or } 1 & \text{o.w.} \cdots (c) \end{cases}$$

because if (a) but $a_n = 0$ then even if $a_m = 1$ for $\forall m \geq n+1$,

$\sum a_n x^n < 1$

if (b) but $a_n = 1$ then $\sum_{m=1}^{\infty} a_m x^m \geq \sum_{m=1}^{n} a_m x^m > 1$

and with this rule, $\sum a_n x^n = 1$ because

i) # of 0's is infinite

for large N, $\exists n \geq N$ st $a_n = 0$ (i.e, (a) does not hold) and

$\varepsilon > \frac{x^{n+1}}{1-x} = x^{n+1} + x^{n+2} + \cdots \geq 1 - (a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1})$, $\forall \varepsilon > 0$

ii) # of 0's is finite

If there are no 0, $x = \frac{1}{2}$, contradiction.

Let n be the largest number st $a_n = 0$

$\Rightarrow x^{n+1} + x^{n+2} + \cdots \geq 1 - (a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1})$

$\Rightarrow \sum_{n=1}^{\infty} a_n x^n \geq 1$ $\quad (a_{n+1} = a_{n+2} = \cdots = 1)$

but since $x^m \leq 1 - (a_1 x + \cdots + a_{m-1} x^{m-1})$ (not (b)) for $m \geq n+1$,

$\sum_{n=1}^{\infty} a_n x^n \leq 1 \Rightarrow \sum_{n=1}^{\infty} a_n x^n = 1$

(a), (b) are disjoint since $x^{n+1} + x^{n+2} + \cdots = \frac{x^{n+1}}{1-x} > x^n$ $\quad (x > \frac{1}{2})$

If (c), we have different sequence for the same x so it has to be removed.

We require that $x \in (\frac{1}{2}, \frac{-1+\sqrt{5}}{2})$ i.e, $x^2 + x < 1$

a_1 must be 1 since $x^2 + x^3 + \cdots = \frac{x^2}{1-x} < 1$, a_2 must be 1 $(x + x^2 < 1)$

Define $b_n :=$ the root of $b_n(x) = x + x^2 + a_3 x^3 + \cdots + a_{n-1} x^{n-1} + x^{n+1} + x^{n+2} + \cdots - 1 = 0$

$c_n :=$ the root of $c_n(x) = x + x^2 + a_3 x^3 + \cdots + a_{n-1} x^{n-1} + x^n - 1 = 0$ $\quad (n \geq 2)$

by the intermediate value thm, b_n, c_n exists and unique (monotone)

we can see that $b_n > c_n$ iff $b_n(x) < c_n(x)$ for $\forall x \in (\frac{1}{2}, \frac{-1+\sqrt{5}}{2})$ and conversely.

So, $b_n < c_n$ $\quad (b_n(x) - c_n(x) = -x^n + x^{n+1} + x^{n+2} + \cdots > 0$ since $x > \frac{1}{2})$

the condition (c) is exactly $x \in [b_n, c_n]$, (a) is $x < b_n$, (b) is $x > c_n$

So, if $a_n = 0$, we require $x > c_n$, if $a_n = 1$, we require $x < b_n$.

for convienence, let $c_1 = \frac{1}{2}$.

Since $a_2 = 1$, $x \in (c_1, b_2)$

Assume that we chose a_n for $\forall n \leq N$

i) $a_N = 1$

for x to be in S, $x \in (c_{N-k}, b_N)$ for some $k \geq 1$

we can choose $a_{N+1} = 0$ with $x \in S$ because $c_{N+1} < b_N$

$$\left(c_{N+1}(x) > b_N(x) \iff x + x^2 + a_3x^3 + \cdots + a_{N-1}x^{N-1} + x^N + x^{N+1} \right.$$
$$> x + x^2 + a_3x^3 + \cdots + a_{N-1}x^{N-1} + x^{N+1} + x^{N+2} + \cdots$$
$$\left. \iff x^N > x^{N+2} + x^{N+3} + \cdots \iff x^N > \frac{x^{N+2}}{1-x} \iff 1 > x^2 + x \right)$$

$\Rightarrow x \in (c_{N+1}, b_N)$

then, we can have two possibilities for a_{N+2} with $x \in S$,

because $c_{N+1} < b_{N+2}$, $c_{N+2} < b_N$

$$\left(c_{N+1}(x) > b_{N+2}(x) \iff x + x^2 + a_3x^3 + \cdots + a_{N-1}x^{N-1} + x^N + x^{N+1} \right.$$
$$> x + x^2 + a_3x^3 + \cdots + a_{N-1}x^{N-1} + x^N + x^{N+3} + x^{N+4} + \cdots$$
$$\iff x^{N+1} > x^{N+3} + x^{N+4} + \cdots \iff 1 > x^2 + x$$

$$c_{N+2}(x) > b_N(x) \iff x + x^2 + a_3x^3 + \cdots + a_{N-1}x^{N-1} + x^N + x^{N+2}$$
$$> x + x^2 + a_3x^3 + \cdots + a_{N-1}x^{N-1} + x^{N+1} + x^{N+2} + \cdots$$
$$\iff x^N > x^{N+1} + x^{N+3} + x^{N+4} + \cdots$$
$$\left. \iff 1 > x + x^3 + x^4 + \cdots \iff x < b_2 \right)$$

ii) $a_N = 0$

for x to be in S, x has to be in (c_N, b_{N-k}) for some $k \geq 1$

we can choose $a_{N+1} = 1$ with $x \in S$ because $c_N < b_{N+1}$

$$\left(c_N(x) > b_{N+1}(x) \iff x + x^2 + a_3x^3 + \cdots + a_{N-1}x^{N-1} + x^N \right.$$
$$> x + x^2 + a_3x^3 + \cdots + a_{N-1}x^{N-1} + x^{N+2} + x^{N+3} + \cdots$$
$$\left. \iff x^N > x^{N+2} + x^{N+3} + \cdots \right)$$

$\Rightarrow x \in (c_N, b_{N+1})$

then, we can have two possibilities for a_{N+2} with $x \in S$,

because $c_N < b_{N+2}$, $c_{N+2} < b_{N+1}$

$$\left(c_N(x) > b_{N+2}(x) \iff x + x^2 + a_3x^3 + \cdots a_{N-1}x^{N-1} + x^N > x + x^2 + a_3x^3 + \cdots + a_{N-1}x^{N-1} \right.$$
$$+ x^{N+1} + x^{N+3} + x^{N+4} + \cdots$$

$\Leftrightarrow x^N > x^{N+1} + x^{N+3} + x^{N+4} + \cdots$

$c_{N+2}(x) > b_{N+1}(x) \Leftrightarrow x + x^2 + a_3 x^3 + \cdots + a_{N-1}x^{N-1} + x^{N+1} + x^{N+2}$

$> x + x^2 + a_3 x^3 + \cdots + a_{N-1}x^{N-1} + x^{N+2} + x^{N+3} + \cdots$

$\Leftrightarrow x^{N+1} > x^{N+3} + x^{N+4} + \cdots$)

By above, there are infinitely many n's s.t a_n can be 0 and can be 1 regardless of prior terms.

$\Rightarrow$ # of $\{a_n\}$'s constructed in this way $\overset{\text{bijective}}{\simeq} \{\{b_n\} \mid b_n = 0 \text{ or } 1\}$ which is uncountable.

It remains to show that # of open intervals to which x has to be restricted in the constructing process (call it I_n) has nonempty intersection (i.e, $\bigcap_{n=1}^{\infty} I_n \neq \phi$) is uncountable.

$\bigcap_{n=1}^{\infty} \bar{I}_n \neq \phi \quad (\bar{I}_1 \supset \bar{I}_2 \supset \cdots)$

If $\bar{x} \in \bigcap_{n=1}^{\infty} \bar{I}_n - \bigcap_{n=1}^{\infty} I_n \subset \bigcup_{n=1}^{\infty} \partial I_n \Rightarrow \bar{x} \in \partial I_N$ for some N i.e, $\bar{x} = b_N$ or c_N

($\sum_{n=1}^{\infty} a_n \bar{x}^n = 1$ because clearly, each $\bar{I}_n$ is an appropriate inteval to choose a_n in constructing)

If $\bar{x} = b_N$, which is a root of $\frac{x^{n+1}}{1-x} = 1 - (x + x^2 + a_3 x^3 + \cdots + a_{n-1}x^{n-1})$

$\Leftrightarrow x^{n+1} - (1 - (x + x^2 + \cdots + a_{n-1}x^{n-1}))(1-x) = 0$

$\bar{x}$ is an algebraic integer.

If $\bar{x} = c_N$, which is a root of $x^n + a_{n-1}x^{n-1} + \cdots + a_3 x^3 + x^2 + x - 1 = 0$

$\bar{x}$ is an algebraic integer.

Since the set of algebraic integer is countable, if we remove all $\{a_n\}$'s corresponding an algebraic integer from all $\{a_n\}$'s constructed, remaning $\{a_n\}$'s has corresponding I_n which satisfies $\bigcap_{n=1}^{\infty} I_n \neq \phi$ and is uncountable, so we are done.