<POW 2010-6> 김치헌 (06학번)

문제 : $\displaystyle\sum_{m=0}^{n}\binom{n}{m}\sum_{k=0}^{m}\binom{m}{k}^3 = \sum_{m=0}^{n}\binom{n}{m}^2\binom{2m}{m}$.

Consider $(1+(1+x)(1+y)(1+z))^n = f(x,y,z)$.

Denote the coefficient of $x^a y^b z^c$ in $f(x,y,z)$ as $f(x,y,z)[x^a y^b z^c]$.

Then, $\displaystyle\sum_{m=k}^{n}\binom{n}{m}\binom{m}{k}^3 = f(x,y,z)[x^k y^k z^k]$.

So, $\displaystyle\sum_{m=0}^{n}\binom{n}{m}\sum_{k=0}^{m}\binom{m}{k}^3 = f(x,y,\tfrac{1}{xy})[1]$.

By the way, $1+(1+x)(1+y)(1+\frac{1}{xy}) = 3+x+y+xy+\frac{1}{x}+\frac{1}{y}+\frac{1}{xy}$

$$= \left(1+x+\frac{1}{y}\right)\left(1+y+\frac{1}{x}\right)$$

$$= \left(1+x\left(1+\frac{1}{xy}\right)\right)\left(1+y\left(1+\frac{1}{xy}\right)\right).$$

So, $f(x,y,\frac{1}{xy}) = \left(1+x\left(1+\frac{1}{xy}\right)\right)^n\left(1+y\left(1+\frac{1}{xy}\right)\right)^n$.

$\Rightarrow$ $\displaystyle f(x,y,\tfrac{1}{xy})[1] = \sum_{k,l}\binom{n}{k}\binom{n}{l}\left[x^k\left(1+\frac{1}{xy}\right)^k y^l\left(1+\frac{1}{xy}\right)^l\right][1]$.

Note that $\left(x^k\left(1+\frac{1}{xy}\right)^k y^l\left(1+\frac{1}{xy}\right)^l\right)[1] = 0$ if $k \neq l$.

($\because$ To cancel $x^k y^l$ by $\left(\frac{1}{xy}\right)^m$ for some m, we need $k=l$).

And, $\left(x^m\left(1+\frac{1}{xy}\right)^m y^m\left(1+\frac{1}{xy}\right)^m\right)[1] = \left(1+\frac{1}{xy}\right)^{2m}[x^m y^m]$

$$= \binom{2m}{m}.$$

So, $\displaystyle f(x,y,\tfrac{1}{xy})[1] = \sum_{m}\binom{n}{m}^2\binom{2m}{m}$.

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