Problem of the Week / 2010-5 김치헌(06학번)

Let y be a rational number s.t. $\cos\pi y$, $\sin\pi y$, 1 are linearly dependent over $\mathbb{Q}$, that is,

$\exists p, q, r \in \mathbb{Q}$ s.t. $p\cos\pi y + q\sin\pi y + r = 0$.

$$\Rightarrow \quad p\cos\pi y + q\sqrt{1-\cos^2\pi y} + r = 0$$

$$\Rightarrow \quad (p\cos\pi y + r)^2 = q^2(1-\cos^2\pi y)$$

$$\Rightarrow \quad \begin{cases} (p^2+q^2)\cos^2\pi y + 2pr\cos\pi y + (r^2-q^2) = 0 \\ (p^2+q^2)\sin^2\pi y + 2qr\sin\pi y + (r^2-p^2) = 0 \end{cases}$$

Define $\eta = e^{\pi i y} = \cos\pi y + i\sin\pi y$.

If $\frac{y}{2} = \frac{n}{m}$ where $\gcd(m,n)=1$, then η is an mth primitive root of unity, i.e., $\eta^m = 1$, and $\eta^k \neq 1$ $\forall k \in \{1, 2, \dots, m-1\}$.

Note that $\cos\pi y = \frac{1}{2}\left(\eta + \frac{1}{\eta}\right)$.

$$\Rightarrow \quad (p^2+q^2)\frac{1}{4}\left(\eta^2 + \frac{1}{\eta^2} + 2\right) + 2pr\cdot\frac{1}{2}\left(\eta + \frac{1}{\eta}\right) + (r^2-q^2) = 0$$

$$\Rightarrow \quad \left(\frac{p^2+q^2}{4}\right)\eta^2 + (pr)\eta + \left(\frac{p^2}{2} - \frac{q^2}{2} + r^2\right) + (pr)\frac{1}{\eta} + \frac{p^2+q^2}{4}\frac{1}{\eta^2} = 0$$

Let $f(x) = \frac{p^2+q^2}{4}x^4 + prx^3 + \left(\frac{p^2}{2} - \frac{q^2}{2} + r^2\right)x^2 + prx + \frac{p^2+q^2}{4}$.

Then $f \in \mathbb{Q}[x]$ and $\deg f = 4$, $f(\eta) = 0$.

Since $f(\eta)=0$ and $\eta^n=1$, we have $f(x) \mid x^m-1$.

And, $f(x) \nmid x^k-1 \quad \forall k\in\{1,2,\dots,m-1\}$. ($\because \eta^k \neq 1 \quad \forall k\in\{1,2,\dots,m-1\}$).

Let $\Phi_d(x)$ be the dth cyclotomic polynomial.

(i.e., $\Phi_d(x) = \prod_{\substack{1\le k\le d \\ (k,d)=1}} (x-e^{\frac{2\pi ki}{d}})$)

Then ① $\Phi_d \in \mathbb{Q}[x] \quad \forall d$, ② Φ_d is irreducible over $\mathbb{Q}$,

③ $x^n-1 = \prod_{d\mid n} \Phi_d(x) \quad \forall n$.

<u>Claim</u> $\Phi_m(x) \mid f(x)$.

($\because$ If not, then every irreducible factor of $f(x)$
is the form of $\Phi_d(x)$, d is a proper divisor of n.
But, $\Phi_d(x) \mid x^d-1$ where $d<n$.)

So, $\deg \Phi_m(x) \le \deg f(x) = 4$.

$\Rightarrow$ <u>$\varphi(m) \le 4$.</u>

① $\varphi(m)=1 \Rightarrow m=1,2$ and $y=1,2$ ($\because \frac{y}{2}=\frac{n}{m}$ and $e^{\pi i y}$ is an mth primitive root of 1)

② $\varphi(m)=2 \Rightarrow m=3,4,6$ and $y=\frac{2}{3},\frac{4}{3},\frac{1}{2},\frac{3}{2},\frac{1}{3},\frac{5}{3}$

③ $\varphi(m)=3 \Rightarrow$ no such m

④ $\varphi(m)=4 \Rightarrow m=5,8,10,12$ and

$y=\frac{2}{5},\frac{4}{5},\frac{6}{5},\frac{8}{5},\frac{1}{4},\frac{3}{4},\frac{5}{4},\frac{7}{4},\frac{1}{5},\frac{3}{5},\frac{7}{5},\frac{9}{5},\frac{1}{6},\frac{5}{6},\frac{7}{6},\frac{11}{6}$.

But, $\sin\frac{\pi}{5} = \sqrt{\frac{5-\sqrt{5}}{8}}$ and $\nexists p, q, r \in \mathbb{Q}$ s.t.

$(p^2+q^2)\sin^2\frac{\pi}{5} + 2qr\sin\frac{\pi}{5} + r^2 - p^2 = 0$.

Similar for $y = \frac{2}{5}, \frac{3}{5}, \cdots, \frac{9}{5}$.

You may verify that $\cos\pi y$, $\sin\pi y$, 1 are linearly indep.

for other cases.

Hence, $y \in \{1, 2, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{1}{6}, \frac{5}{6}, \frac{7}{6}, \frac{11}{6}\}$.

Simply, $y \in \{\frac{k}{12} \mid 1 \le k \le 24,$ k and 12 are not coprime$\}$.

Remark:
Observe that if y is a solution,
then y+2 is a solution as well.
So the set of solutions should be:
{ k/12 : gcd(k,12) is not 1, k: integer}