< POW − 2010 − 4 > 김치헌 (06학번)

Note that $f_k(n) := \sum_{i=1}^{n} k^{\gcd(i,n)} = \sum_{d|n} \#\{1 \le i \le n \mid d = \gcd(i,n)\} k^d$

$$= \sum_{d|n} \varphi\left(\frac{n}{d}\right) k^d \left(= \sum_{d|n} \varphi(d) k^{\frac{n}{d}} \right).$$

let p be a prime divisor of n.

Then $n = p^s \cdot m$ for some $s \ge 1$, m and p are coprime.

$$\Rightarrow f_k(n) = \sum_{d|n} \varphi(d) k^{\frac{n}{d}}$$

$$= \sum_{e|m} \sum_{t=0}^{s} \varphi(p^t e) k^{p^{s-t} \cdot \frac{m}{e}}$$

$$= \sum_{e|m} \varphi(e) \left[\sum_{t=0}^{s} \varphi(p^t) \left(k^{\frac{m}{e}}\right)^{p^{s-t}} \right].$$

To show $p^s \mid f_k(n)$, it is enough to show that

$\forall k$, $\sum_{t=0}^{s} \varphi(p^t) k^{p^{s-t}}$ is divisible by p^s.

Lemma $\forall$ prime p $\forall k, \ell \ge 1$, $k^{p^\ell} \equiv k^{p^{\ell-1}} \pmod{p^\ell}$.

pf) If $p \nmid k$, then $k^{\varphi(p^\ell)} = k^{p^\ell - p^{\ell-1}} \equiv 1 \pmod{p^\ell}$.

(Euler's thm)

If $p \mid k$, then $p^{\ell-1} \ge 2^{\ell-1} \ge \ell$.

So, $p^\ell \mid p^{p^{\ell-1}} \mid k^{p^{\ell-1}} \Rightarrow k^{p^\ell} \equiv k^{p^{\ell-1}} \equiv 0 \pmod{p^\ell}$.

$\therefore$ $k^{p^\ell} \equiv k^{p^{\ell-1}} \pmod{p^\ell}$ for all $k, \ell \ge 1$, prime p. //

Claim $\quad p^s \mid \sum_{t=0}^{s} \varphi(p^t) k^{p^{s-t}} \quad \forall k,\ s \geq 1.$

pf)
$$\sum_{t=0}^{s} \varphi(p^t) k^{p^{s-t}} = k^{p^s} + \sum_{t=1}^{s-1} (p^t - p^{t-1}) k^{p^{s-t}} + (p^s - p^{s-1})k$$
$$\equiv \left(k^{p^s} - p^{s-1}k\right) + \sum_{t=1}^{s-1} \left(p^t k^{p^{s-t}} - p^{t-1} k^{p^{s-t}}\right) \pmod{p^s} \quad \cdots (*)$$

By lemma,
$$\begin{cases} k^{p^s} \equiv k^{p^{s-1}} \pmod{p^s} \\ p^t k^{p^{s-t}} \equiv p^t k^{p^{s-t-1}} \pmod{p^s}. \end{cases}$$

$$\Rightarrow (*) \equiv \left(k^{p^{s-1}} - p^{s-1}k\right) + \sum_{t=1}^{s-1} \left(p^t k^{p^{s-t-1}} - p^{t-1} k^{p^{s-t}}\right) \pmod{p^s}$$
$$= \left(k^{p^{s-1}} - p^{s-1}k\right) + \sum_{t=1}^{s-1} p^t k^{p^{s-t-1}} - \sum_{t=0}^{s-2} p^t k^{p^{s-t-1}}$$
$$= \left(k^{p^{s-1}} - p^{s-1}k\right) + \left(p^{s-1}k - k^{p^{s-1}}\right).$$
$$= 0$$

$$\therefore \quad \sum_{t=0}^{s} \varphi(p^t) k^{p^{s-t}} \equiv 0 \pmod{p^s} \quad \forall k,\ s \geq 1.$$

By the Claim, $\quad p^s \mid f_k(n)$. for any $p^s \| n$.

$$\therefore \quad n \mid f_k(n), \text{ i.e.,}$$

$\sum_{i=1}^{n} k^{\gcd(i,n)}$ is divisible by n.

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