POW 2010-1 Covering the unit square by squares

Let $x_1, \cdots, x_n$ be length of each square s.t $x_1 \geq \cdots \geq x_n$, $A_1, \cdots, A_n$ be corresponding squares

WLOG, we may assume $x_i < 1$ (If $x_i \geq 1 \Rightarrow$ cover using A_i)

We will cover the unit square using this method :

i) choose n_1 s.t $x_1 + \cdots + x_{n_1} \geq 1$, $x_1 + \cdots + x_{n_1 - 1} < 1$,

(this is possible because $x_1 + \cdots + x_n \geq x_1^2 + \cdots + x_n^2 \geq 3$)

choose n_2 s.t $x_{n_1+1} + \cdots + x_{n_1+n_2} \geq 1$, $x_{n_1+1} + \cdots + x_{n_1+n_2-1} < 1$,

continue this process until it's impossible.

(For convenience rename $x_1, \cdots, x_n$ as $x_{1,1}, \cdots, x_{1,n_1}, x_{2,1}, \cdots, x_{2,n_2}, \cdots,$ $x_{m,1}, \cdots, x_{m,n_m}$ and similarly for $A_1, \cdots, A_n$)

ii) Attach $A_{i,1}, \cdots, A_{i,n_i}$ horizontally for each i

then $R_i \subset A_{i,1} \cup \cdots \cup A_{i,n_i}$, where R_i is the rectangle with width 1, height x_{i,n_i}.

Regard $A_{i,1} \cup \cdots \cup A_{i,n_i}$ as R_i and attach R_i vertically.

We will prove that $R_1 \cup \cdots \cup R_m$ covers the unit square

i) $x_{m,1} + \cdots + x_{m,n_m} < 1$

It suffices to show that $x_{1,n_1} + \cdots + x_{m-1,n_{m-1}} \geq 1$

we have $1 > x_{i,1} + \cdots + x_{i,n_i-1}$ $\quad (2 \leq i \leq m-1)$

$$\Rightarrow x_{i-1,n_{i-1}} > x_{i-1,n_{i-1}}(x_{i,1} + \cdots + x_{i,n_i-1}) \qquad (\because x_1 \geq x_2 \geq \cdots \geq x_n)$$

$$\geq x_{i,1}^2 + \cdots + x_{i,n_i-1}^2 \qquad \cdots (a)$$

also, $1 > x_{m,1} + \cdots + x_{m,n_m}$

$$\Rightarrow x_{m-1,n_{m-1}} > x_{m-1,n_{m-1}}(x_{m,1} + \cdots + x_{m,n_m})$$

$$\geq x_{m,1}^2 + \cdots + x_{m,n_m}^2 \qquad \cdots (b)$$

sum (a)'s and (b):

$$x_{1,n_1} + \cdots + x_{m-1,n_{m-1}} \geq (x_1^2 + \cdots + x_n^2) - (x_{1,1}^2 + \cdots + x_{1,n_1-1}^2) - (x_{1,n_1}^2 + \cdots + x_{m-1,n_{m-1}}^2)$$

(But, $x_1^2 + \cdots + x_n^2 \geq 3$, $1 > x_{1,1} + \cdots + x_{1,n_1-1} \geq x_{1,1}^2 + \cdots + x_{1,n_1-1}^2$ $(\because x_i < 1)$

and $x_{1,n_1}^2 + \cdots + x_{m-1,n_{m-1}}^2 \leq x_{1,n_1} + \cdots + x_{m-1,n_{m-1}}$)

$$x_{1,n_1} + \cdots + x_{m-1,n_{m-1}} \geq 3 - 1 - (x_{1,n_1} + \cdots + x_{m-1,n_{m-1}})$$

$$\Rightarrow x_{1,n_1} + \cdots + x_{m-1,n_{m-1}} \geq 1$$

ii) $x_{m,1}+\cdots+x_{m,n_m}\geq 1$

It suffices to show that $x_{1,n_1}+\cdots+x_{m,n_m}\geq 1$

By definition of m (the final index), $x_{m,1}+\cdots+x_{m,n_m-1}<1$

we have $1>x_{i,1}+\cdots+x_{i,n_i-1}$ $(2\leq i\leq m)$

$$\Rightarrow x_{i-1,n_{i-1}}>x_{i-1,n_{i-1}}(x_{i,1}+\cdots+x_{i,n_i-1})$$
$$\geq x_{i,1}^2+\cdots+x_{i,n_i-1}^2$$

By summing these, we have

$$x_{1,n_1}+\cdots+x_{m-1,n_{m-1}}\geq(x_1^2+\cdots+x_n^2)-(x_{1,1}^2+\cdots+x_{1,n_1-1}^2)-(x_{1,n_1}^2+\cdots+x_{m,n_m}^2)$$

since $x_{m,n_m}\geq x_{m,n_m}^2$,

$$x_{1,n_1}+\cdots+x_{m,n_m}\geq(x_1^2+\cdots+x_n^2)-(x_{1,1}^2+\cdots+x_{1,n_1-1}^2)-(x_{1,n_1}^2+\cdots+x_{m-1,n_{m-1}}^2)$$

$$\left(\begin{array}{l} x_1^2+\cdots+x_n^2\geq 3,\ 1>x_{1,1}+\cdots+x_{1,n_1-1}\geq x_{1,1}^2+\cdots+x_{1,n_1-1}^2 \\ \text{and } x_{1,n_1}^2+\cdots+x_{m-1,n_{m-1}}^2\leq x_{1,n_1}+\cdots+x_{m-1,n_{m-1}}+x_{m,n_m}\end{array}\right)$$

$$\Rightarrow x_{1,n_1}+\cdots+x_{m,n_m}\geq 3-1-(x_{1,n_1}+\cdots+x_{m,n_m})$$

$$\Rightarrow x_{1,n_1}+\cdots+x_{m,n_m}\geq 1$$

$\therefore$ proved

Since $R_1\cup\cdots\cup R_m$ covers the unit square,

$A_1\cup\cdots\cup A_n\supset R_1\cup\cdots\cup R_m$ (after attaching),

$A_1\cup\cdots\cup A_n$ covers the unit square.