POW 2009-22 Integral and Limit

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since $0 \le t \le 2\varepsilon$, $\varepsilon \to 0$

$\Rightarrow \sin t \to t$, $\sin\varepsilon \to \varepsilon$

$$\lim_{\varepsilon\to0}\int_0^{2\varepsilon}\log\left(\frac{|\sin t-\varepsilon|}{\sin\varepsilon}\right)\frac{dt}{\sin t} = \lim_{\varepsilon\to0}\int_0^{2\varepsilon}\log\left(\frac{|t-\varepsilon|}{\varepsilon}\right)\frac{dt}{t} \quad \cdots (*)$$

$$= \lim_{\varepsilon\to0}\left(\int_0^{\varepsilon}\log\left(1-\frac{t}{\varepsilon}\right)\frac{dt}{t} + \int_{\varepsilon}^{2\varepsilon}\log\left(\frac{t}{\varepsilon}-1\right)\frac{dt}{t}\right)$$

$$= \int_0^1 \log(1-x)\frac{dx}{x} + \int_1^2 \log(x-1)\frac{dx}{x} \qquad (\text{substitute } x = \frac{t}{\varepsilon})$$

$$\int_0^1 \log(1-x)\frac{dx}{x} = \int_0^\infty \frac{te^{-t}}{1-e^{-t}}dt \qquad (x = 1-e^{-t})$$

$$= \int_0^\infty t\left(\sum_{k=1}^\infty e^{-kt}\right)dt \qquad (e^{-t}<1 \text{ for } t>0)$$

$$= t\sum_{k=1}^\infty\left(-\frac{1}{k}e^{-kt}\right)\Big|_0^\infty + \int_0^\infty \sum_{k=1}^\infty\left(\frac{1}{k}e^{-kt}\right)dt$$

$$= \int_0^\infty \sum_{k=1}^\infty \frac{1}{k}e^{-kt}dt \qquad \left(\because t\sum_{k=1}^\infty\left(\frac{1}{k}e^{-kt}\right) \le t\sum_{k=1}^\infty e^{-kt} = \frac{te^{-t}}{1-e^{-t}} \to 0 \text{ as } t\to\infty\right)$$

$$= -\sum_{k=1}^\infty \frac{1}{k^2}e^{-kt}\Big|_0^\infty = -\sum_{k=1}^\infty\frac{1}{k^2} = -\frac{\pi^2}{6}$$

$$\int_1^2 \log(x-1)\frac{dx}{x} = \int_0^\infty \frac{-te^{-t}}{1+e^{-t}}dt \qquad (x = 1+e^{-t})$$

$$= \int_0^\infty t\left(\sum_{k=1}^\infty (-1)^k e^{-kt}\right)dt$$

$$= t\sum_{k=1}^\infty\left(-\frac{1}{k}(-1)^k e^{-kt}\right)\Big|_0^\infty + \int_0^\infty \sum_{k=1}^\infty\left(-\frac{1}{k}(-1)^k e^{-kt}\right)dt$$

$$= \int_0^\infty \sum_{k=1}^\infty\left(-\frac{1}{k}(-1)^k e^{-kt}\right)dt$$

$$= \sum_{k=1}^\infty \frac{1}{k^2}(-1)^k e^{-kt}\Big|_0^\infty = -\sum_{k=1}^\infty\frac{(-1)^{k+1}}{k^2} = -\frac{\pi^2}{12}$$

$\therefore -\frac{\pi^2}{4}$

($(*)$: $\displaystyle\int_0^{2\varepsilon}\log\left(\frac{|\sin t-\varepsilon|}{\sin\varepsilon}\right)\frac{dt}{\sin t} = \int_0^{\sin(2\varepsilon)}\log\left(\frac{|x-\varepsilon|}{\sin\varepsilon}\right)\frac{dx}{x\sqrt{1-x^2}}$ $\qquad (x = \sin t)$

since $\frac{1}{x}\left(1-\frac{1}{\sqrt{1-x^2}}\right)$ is bdd at $0\le x\le \sin(2\varepsilon)$, $\to 0$ as $\varepsilon\to0$,

(consider $\left(\frac{1}{\sqrt{1-x^2}}\right)'$ and MVT)

also $\displaystyle\int_0^{\sin(2\varepsilon)}\log\left(\frac{|x-\varepsilon|}{\sin\varepsilon}\right)dx$ converges, $\displaystyle\left|\int_0^{\sin(2\varepsilon)}\log\left(\frac{|x-\varepsilon|}{\sin\varepsilon}\right)\frac{dx}{x} - \int_0^{\sin(2\varepsilon)}\log\left(\frac{|x-\varepsilon|}{\sin\varepsilon}\right)\frac{dx}{x\sqrt{1-x^2}}\right| \to 0$

as $\varepsilon\to0$. $\Rightarrow \displaystyle\lim_{\varepsilon\to0}\int_0^{2\varepsilon}\log\left(\frac{|\sin t-\varepsilon|}{\sin\varepsilon}\right)\frac{dt}{\sin t} = \lim_{\varepsilon\to0}\int_0^{\sin(2\varepsilon)}\log\left(\frac{|x-\varepsilon|}{\sin\varepsilon}\right)\frac{dx}{x}$ $\cdots$ (a)

$\log\left(\frac{|x-\varepsilon|}{\sin\varepsilon}\right)\frac{1}{x} - \log\left(\frac{|x-\varepsilon|}{\varepsilon}\right)\frac{1}{x} = \log\left(\frac{\varepsilon}{\sin\varepsilon}\right)\frac{1}{x}$,

$\displaystyle\lim_{\varepsilon\to0}\int_0^{\sin(2\varepsilon)}\log\left(\frac{\varepsilon}{\sin\varepsilon}\right)\frac{dx}{x} \to 0$ as $\varepsilon\to0$ (we think the integral like $\displaystyle\lim_{\varepsilon\to0}\int_{\varepsilon^2}^{\sin(2\varepsilon)}$ because otherwise diverges)

$\Rightarrow \displaystyle\lim_{\varepsilon\to0}\int_0^{\sin(2\varepsilon)}\log\left(\frac{|x-\varepsilon|}{\sin\varepsilon}\right)\frac{dx}{x} = \lim_{\varepsilon\to0}\int_0^{\sin(2\varepsilon)}\log\left(\frac{|x-\varepsilon|}{\varepsilon}\right)\frac{dx}{x} = \lim_{\varepsilon\to0}\int_0^{2\varepsilon}\log\left(\frac{|x-\varepsilon|}{\varepsilon}\right)\frac{dx}{x}$ $\cdots$ (b)

since $\displaystyle\int_{\sin(2\varepsilon)}^{2\varepsilon}\log\left(\frac{|x-\varepsilon|}{\varepsilon}\right)\frac{dx}{x} \to 0$ as $\varepsilon\to0$

(a) and (b) gives the result)