POW 2009-21 Rank and Eigenvalues

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Let $\varepsilon = e^{i\theta}$, $x_\ell = (1\ \varepsilon^\ell\ \varepsilon^{2\ell} \cdots \varepsilon^{(n-1)\ell})^T$ for $0 \le \ell \le n-1$

and $\lambda_\ell = 1 + \varepsilon^\ell \cos\theta + \varepsilon^{2\ell}\cos 2\theta + \cdots + \varepsilon^{(n-1)\ell}\cos(n-1)\theta$

then, $Ax_\ell = \lambda_\ell x_\ell$

Let $B = (x_0\ x_1\ \cdots\ x_{n-1})$

since $\mathrm{rank}(B^2) \le \mathrm{rank}(B)$, B^2 is invertible $\Rightarrow$ B is invertible

$B^2 = (\varepsilon^{ij})_{0\le i,j\le n-1} (\varepsilon^{ij})_{0\le i,j\le n-1} = \left(\sum_{k=0}^{n-1} \varepsilon^{(i+j)k}\right)_{0\le i,j\le n-1}$

If $\varepsilon^{(i+j)} \ne 1$, $\sum_{k=0}^{n-1} \varepsilon^{(i+j)k} = \frac{1-\varepsilon^{(i+j)n}}{1-\varepsilon^{i+j}} = 0$ $\quad (\because \varepsilon^n = 1)$

If $\varepsilon^{(i+j)} = 1$, $\sum_{k=0}^{n-1} \varepsilon^{(i+j)k} = n$

since $0 \le i, j \le n-1$, $\varepsilon^{(i+j)} = 1$ iff $n \mid i+j$

$i = 0 \Rightarrow \varepsilon^{(i+j)} = 1$ iff $j = 0$

$i > 0 \Rightarrow \varepsilon^{(i+j)} = 1$ iff $j = n - i$

$\Rightarrow B^2 = \begin{pmatrix} n & 0 & \cdots & 0 & 0 \\ 0 & & & 0 & n \\ 0 & & 0 & n & 0 \\ \vdots & & \iddots & & \\ 0 & n & & & 0 \end{pmatrix}$ is invertible, so is B

$\Rightarrow \{x_0, \cdots, x_{n-1}\}$ is a linearly independent set

$\Rightarrow \lambda_0, \cdots, \lambda_{n-1}$ are all eigenvalues, $\mathrm{nullity}(A) = \#$ of ℓ s.t. $\lambda_\ell = 0$

Since A is symmetric and real, λ_ℓ's are real

$(\because Ax = \lambda x \Rightarrow x^T A^T = x^T A = \lambda x^T \Rightarrow \bar{x}^T A = \bar{\lambda}\bar{x}^T$

$(\bar{x}^T A)x = \bar{\lambda}\bar{x}^T x$, $\bar{x}^T(Ax) = \lambda \bar{x}^T x \Rightarrow \bar{\lambda}\|x\|^2 = \lambda\|x\|^2$, $x \ne 0 \Rightarrow \bar{\lambda} = \lambda$)

$\Rightarrow \lambda_\ell = \sum_{k=0}^{n-1} \cos k\theta\, e^{k\ell\theta i} = \sum_{k=0}^{n-1} \cos k\theta \cos k\ell\theta$

$= \sum_{k=0}^{n-1} \frac{1}{2}(\cos k(\ell+1)\theta + \cos k(\ell-1)\theta)$

(for $n \nmid m$, $\sum_{k=0}^{n-1} \cos km\theta = \mathrm{Re}\left(\sum_{k=0}^{n-1} e^{km\theta i}\right) = \mathrm{Re}\left(\frac{1-e^{nm\theta i}}{1-e^{m\theta i}}\right) = 0$ $\quad (e^{nm\theta i} = 1)$

for $n \mid m$, $\sum_{k=0}^{n-1} \cos km\theta = n$)

Hence, if $n \ge 3$, $\lambda_\ell = \begin{cases} \frac{n}{2} & \text{if } \ell = 1 \text{ or } n-1 \quad (1 \ne n-1) \\ 0 & \text{otherwise} \end{cases}$ $\quad (\because \ell = 1 \Rightarrow n \mid \ell-1,\ \ell = n-1 \Rightarrow n \mid \ell+1)$

if $n = 2$, $\lambda_\ell = \begin{cases} n = 2 & \text{if } \ell = 1 \\ 0 & \text{if } \ell = 0 \end{cases}$

$\mathrm{rank}(A) = n - \mathrm{nullity}(A) = \begin{cases} n-(n-2) = 2 & \text{if } n \ge 3 \\ 2-1 = 1 & \text{if } n = 2 \end{cases}$

$\therefore$ If $n = 2$, $\mathrm{rank}(A) = 1$, eigenvalues : $0, 2$

If $n \ge 3$, $\mathrm{rank}(A) = 2$, eigenvalues : $0, \frac{n}{2}$