POW 2009-10 Expectation

정성구
sg.jeong054@gmail.com

$$e_n = \int_0^1\int_{x_1}^1\int_{x_2}^1 \cdots \int_{x_{n-1}}^1 \frac{x_1x_2\cdots x_n}{(1-x_1)\cdots(1-x_{n-1})}\, dx_n dx_{n-1}\cdots dx_1 \quad \cdots (*)$$

$$= \int_0^1\int_0^{y_1}\int_0^{y_2}\cdots\int_0^{y_{n-1}} \frac{(1-y_1)(1-y_2)\cdots(1-y_n)}{y_1y_2\cdots y_{n-1}}\, dy_n dy_{n-1}\cdots dy_1 \quad (\text{substitute } y_i = 1-x_i)$$

Let $f_1(y_1)=1$,

$$f_n(y_1) = \frac{1}{y_1}\int_0^{y_1}\frac{1-y_2}{y_2}\int_0^{y_2}\cdots\int_0^{y_{n-1}}(1-y_n)dy_n\cdots dy_2 \quad (n\ge 2)$$

then, $e_n = \int_0^1 (1-y_1)f_n(y_1)dy_1$

$$= \left[(1-y_1)\int_0^{y_1}f_n(\xi)d\xi\right]_0^1 + \int_0^1\int_0^{y_1}f_n(\xi)d\xi\, dy_1$$

$$= \int_0^1\int_0^{y_1}f_n(\xi)d\xi dy_1 \quad (\text{first term is zero})$$

f_n has following recurrence relation ;

$$f_{n+1}(y_1) = \frac{1}{y_1}\int_0^{y_1}(1-\xi)f_n(\xi)d\xi \quad \cdots ①$$

($\{e_i\}$ is a cauchy seq. implies $\{f_i\}$ is cauchy, since $f_n\ge 0$, $e_n = \int_0^1\int_0^{y_1}f_n(\xi)d\xi dy_1$

hence, $f_n \to f$ uniformly)

$$\lim_{n\to\infty} y_1 f_{n+1}(y_1) = y_1 f(y_1) = \lim_{n\to\infty}\int_0^{y_1}(1-\xi)f_n(\xi)d\xi = \int_0^{y_1}(1-\xi)f(\xi)d\xi$$

$$\Rightarrow (y_1f(y_1))' = y_1f'(y_1)+f(y_1) = (1-y_1)f(y_1)$$

$$\Rightarrow f'(y_1)+f(y_1)=0, \quad f(y_1)=ce^{-y_1}$$

By ①, $f_1(0)=1$, $\lim_{y_1\to 0} f_{n+1}(y_1) = \lim_{y_1\to 0}\frac{1}{y_1}\int_0^{y_1}(1-\xi)f_n(\xi)d\xi = \lim_{y_1\to 0} f_n(y_1)$

$\Rightarrow \lim_{y_1\to 0} f_n(y_1) = 0$ for $\forall n$, $\lim_{y_1\to 0} f(y_1)=0 \Rightarrow c=1$

$$\lim_{n\to\infty} e_n = \lim_{n\to\infty}\int_0^1\int_0^{y_1}f_n(\xi)d\xi dy_1 = \int_0^1\int_0^{y_1}e^{-\xi}d\xi dy_1$$

$$= \int_0^1(1-e^{-y_1})dy_1 = \frac{1}{e}$$

(The assumption that e_n converges can be proved in this way ;

By direct computation of $(*)$, $e_n = \sum_{x_1=2}^{2}\frac{1}{x_1}\sum_{x_2=2}^{x_1+1}\frac{1}{x_2}\cdots\sum_{x_n=2}^{x_{n-1}+1}\frac{1}{x_n}$

Let $e_{n,k} = \sum_{x_1=k}^{k}\frac{1}{x_1}\sum_{x_2=2}^{x_1+1}\frac{1}{x_2}\cdots\sum_{x_n=2}^{x_{n-1}+1}\frac{1}{x_n}$ $\quad (e_n = e_{n,2})$

then, $e_{n+1,2} = \frac{1}{2}(e_{n,2}+e_{n,3})$, $e_{n+2,3} = \frac{1}{3}(e_{n,2}+e_{n,3}+e_{n,4})$ and so on. $\cdots$ ②

since $e_{1,k} = \frac{1}{k}$ $(k=2,3,\cdots)$, $e_{1,2} > e_{1,3} > e_{1,4} > \cdots$

By ②, if $e_{n,2} > e_{n,3} > \cdots \Rightarrow e_{n+1,2} > e_{n+1,3} > \cdots$

$\Rightarrow e_{n+1,2} = \frac{1}{2}(e_{n,2}+e_{n,3}) < e_{n,2}$ i.e, $e_{n+1} < e_n$ for $\forall n$, $0 \le e_n$.

By Monotone convergence thm, e_n converges)