POW 2009-18 Differential Equation

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LEMMA : Let $R := [a,b]\times[c,d]$ be a rectangular region, $(x_0,y_0)\in R^{\circ}$

If $f(x,y)$, $\partial f/\partial y$ are continuous on R, then there exists some interval $I_0=(x_0-h,\ x_0+h)\subset(a,b)$ s.t if $y_1(x)$, $y_2(x)$ satisfies $\frac{dy}{dx}=f(x,y)$, $y(x_0)=y_0$ on R, then $y_1(x)=y_2(x)$ on I_0

pf) Let $u(x):=y_1(x)-y_2(x)$ on I_0 ($u(x_0)=0$).

for x s.t $0<|x-x_0|<h$, since u is continuous on $[x_0,x]$ (or on $[x,x_0]$), differentiable on (x_0,x) (or on (x,x_0)) we can use MVT

$\frac{u(x)-u(x_0)}{x-x_0}=u'(x_1)=y_1'(x_1)-y_2'(x_1)=f(x_1,y_1(x_1))-f(x_1,y_2(x_1))$ for $x_1\in(x_0,x)$ or (x,x_0)

since y_1,y_2 are continuous, we can choose h small that $c\le y_1(x),y_2(x)\le d$ for $x\in I_0$,

Again, since $f(x_1,y)$ is continuous on $[y_1(x_1),y_2(x_1)]$ or $[y_2(x_1),y_1(x_1)]$, $f_y(x_1,y)$ exists on $(y_1(x_1),y_2(x_1))$ or $(y_2(x_1),y_1(x_1))$, we can use MVT

$f(x_1,y_1(x_1))-f(x_1,y_2(x_1))=f_y(x_1,\tilde{y})(y_1(x_1)-y_2(x_1))=f_y(x_1,\tilde{y})u(x_1)$

Therefore, $|u(x)|=|x-x_0||f_y(x_1,\tilde{y})||u(x_1)|$ $\quad \tilde{y}\in(y_1(x_1),y_2(x_1))$ or $(y_2(x_1),y_1(x_1))$

$\le hM|u(x_1)|$, where $|f_y(x,y)|\le M$ for $\forall(x,y)\in R$

since f_y cont on compact set R

Iterating this $\Rightarrow |u(x)|\le h^nM^n|u(x_n)|\le(hM)^nM_1$, where $|u(x)|\le M_1$ on R by similar reason

Choose h so small that $hM<1$ $\Rightarrow$ $hM\to0$ as $n\to\infty$ $\Rightarrow$ $u(x)=0$ on I_0 $\lrcorner$

Now we find all of the solutions. ($f(x,y)=\sqrt{y}$)

Solving formally, $\frac{dy}{\sqrt{y}}=dx$, $2\sqrt{y}+c=x \Rightarrow y=\frac{(x-c)^2}{4}$ and $y=0$ is also a solution.

Moreover, $\frac{dy}{dx}=\sqrt{y}\ge0$ suggests $y_c:=\begin{cases}\frac{(x-c)^2}{4} & x\ge c\\ 0 & x\le c\end{cases}$ are solutions.

clearly, y_c is C^1, $\frac{dy_c}{dx}=\sqrt{y_c}$ and $y_c(0)=0$ only if $c\ge0$

Claim: 0 and y_c ($c\ge0$) are all solutions.

pf) If $\bar{y}$ is a solution s.t $\bar{y}\ne0$, $\bar{y}\ne y_c$ for $\forall c$,

Since $\bar{y}\ge0$, $\bar{y}'\ge0 \Rightarrow \{x\mid\bar{y}(x)=0\}$ has an upperbound ($\bar{y}\ne0$)

$\bar{y}$ is cont $\Rightarrow$ there is $a:=\sup\{x\mid\bar{y}(x)=0\}$ s.t $\bar{y}(a)=0$ ($\Rightarrow \bar{y}(x)=0$ for $x\le a$, $\bar{y}(x)>0$ for $x>a$)

Since $\{y_c(x)\mid c,x\in\mathbb{R}\}=\{(x,y)\mid y\ge0\}$, $\exists c\in\mathbb{R}$ s.t $\bar{y}(a+1)=y_c(a+1)>0$

Consider $b := \sup\{ k \geq 0 \mid \bar{y}(x) = y_c(x) \text{ for } \forall x \in [a+1-k, a+1+k] \} \geq 0$

$\bar{y} - y_c$ is conti $\Rightarrow$ $y(x) = y_c(x)$ for $x \in [a+1-b, a+1+b]$ (b exists since $y \not\equiv y_c$)

Let $R := [a, a+b+2] \times [\frac{\bar{y}(a+b+1)}{2}, \bar{y}(a+b+1)+1]$ $\Rightarrow$ $f = \sqrt{y}$, $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}}$ are conti on R.

and $\bar{y}, y_c$ are solutions of $\frac{dy}{dx} = \sqrt{y}$, $y(a+b+1) = \bar{y}(a+b+1)$ on R but $\nexists h > 0$

s.t. if $x \in (a+b+1-h, a+b+1+h)$ $\Rightarrow$ $\bar{y} = y_c$ $\Rightarrow$ contradiction $(((a+b+1), \bar{y}(a+b+1)) \in R^\circ)$

$\therefore y = 0$, $y_c = \begin{cases} \frac{(x-c)^2}{4} & x \geq c \\ 0 & x \leq c \end{cases}$