POW2009-15 Double Sum

(0, 1)
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Since $0 \le \lim_{n\to\infty}\sum_{m=1}^{n-1}\frac{1}{nm} \le \lim_{n\to\infty}\frac{1+\log n}{n} = 0$,

$\sum_{n=2}^{\infty}\sum_{m=1}^{n-1}\frac{(-1)^n}{mn}$ converges by alternating series test. Let's denote it by I.

$I = \lim_{z\to\infty}\sum_{n=2}^{z}\sum_{m=1}^{n-1}\frac{(-1)^n}{mn} = \lim_{z\to\infty}\sum_{k=1}^{z-1}\sum_{m=1}^{z-k}\frac{(-1)^{m+k}}{m(m+k)}$ $(k=n-m)$, since

$f: \{(m,n)\in\mathbb{Z}^2 \mid 1\le m\le n-1,\ 2\le n\le z\} \to \{(m,k)\in \mathbb{Z}^2 \mid 1\le m \le z-k,\ 1\le k\le z-1\}$

s.t. $f(m,n) = (m, n-m)$ is bijective.

Lemma1: $\lim_{z\to\infty}\sum_{k=1}^{z-1}\sum_{m=z-k+1}^{\infty}\frac{(-1)^{m+k}}{m(m+k)} = 0$

pf) $\left|\sum_{m=z-k+1}^{\infty}\frac{(-1)^{m+k}}{m(m+k)}\right| \le \frac{1}{(z-k+1)(z+1)}$, since it converges and

If $\sum_{m=0}^{\infty}(-1)^m a_m$ converges, $a_m \downarrow 0$, its subsequence of partial sum $(a_i \ge 0)$

$a_0,\ a_0-(a_1-a_2),\ a_0-(a_1-a_2)-(a_3-a_4),\ \cdots$ is decreasing so

$\left|\sum_{m=0}^{\infty}(-1)^m a_m\right| \le a_0$ (substitute $(z-k+1)$ to m). $\cdots(*)$

$$\lim_{z\to\infty}\left|\sum_{k=1}^{z-1}\sum_{m=z-k+1}^{\infty}\frac{(-1)^{m+k}}{m(m+k)}\right| \le \lim_{z\to\infty}\sum_{k=1}^{z-1}\left|\sum_{m=z-k+1}^{\infty}\frac{(-1)^{m+k}}{m(m+k)}\right|$$

$$\le \lim_{z\to\infty}\sum_{k=1}^{z-1}\frac{1}{(z-k+1)(z+1)}$$

$$= \lim_{z\to\infty}\frac{1}{(z+1)}\sum_{k'=2}^{z}\frac{1}{k'} \qquad (k'=z-k+1)$$

$$\le \lim_{z\to\infty}\frac{\ln z}{z+1} = 0 \qquad \square$$

By above lemma,

$$I = \lim_{z\to\infty}\sum_{k=1}^{z-1}\sum_{m=1}^{z-k}\frac{(-1)^{m+k}}{m(m+k)} + \lim_{z\to\infty}\sum_{k=1}^{z-1}\sum_{m=z-k+1}^{\infty}\frac{(-1)^{m+k}}{m(m+k)} \quad \text{(second term is zero)}$$

$$= \lim_{z\to\infty}\sum_{k=1}^{z-1}\sum_{m=1}^{\infty}\frac{(-1)^{m+k}}{m(m+k)} = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+k}}{m(m+k)}$$

Consider

$\frac{1}{1-x} = \sum_{m=0}^{\infty}x^m$ $(|x|<1)$ integrating 0 to x both sides,

$-\ln(1-x) = \sum_{m=1}^{\infty}\frac{x^m}{m}$ multiplying by x^{k-1} and taking $\sum_{k=1}^{\infty}$ both sides,

$-\frac{\ln(1-x)}{1-x} = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{x^{m+k-1}}{m}$ integrating 0 to x both sides,

$\frac{(\ln(1-x))^2}{2} = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{x^{m+k}}{m(m+k)}$ $(|x|<1)$

since LHS, RHS are continuous on $[-1, 1)$, (because $\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+k}}{m(m+k)}$ converges)

Lemma 2: $\lim_{x\to -1+0}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{x^{m+k}}{m(m+k)} = I$

pf) It suffices to show that $\lim_{x\to -1+0}\left|\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{x^{m+k}-(-1)^{m+k}}{m(m+k)}\right| = 0$ (assume $-1<x<-\frac{1}{2}$)

$$\lim_{x\to-1+0}\left|\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{x^{m+k}-(-1)^{m+k}}{m(m+k)}\right| = \lim_{x\to-1+0}\left|\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{(x-(-1))(m+k)\xi^{k+m-1}}{m(m+k)}\right| \quad (-1<\xi<x) \text{ by Mean Value Theorem}$$

$$= \lim_{x\to-1+0}\left|\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{(x+1)\xi^{k+m-1}}{m}\right|$$

$$\le \lim_{x\to-1+0}\sum_{k=1}^{\infty}\left|\sum_{m=1}^{\infty}\frac{(x+1)\xi^{k+m-1}}{m}\right| \le \lim_{x\to-1+0}\sum_{k=1}^{\infty}\left|(x+1)\xi^{k}\right| = \lim_{x\to-1+0}\left|\frac{(x+1)\xi}{1-\xi}\right| = 0 \qquad \square$$

$\left|\frac{(x+1)\xi^{k+m-1}}{m}\right|$ decreasing as m increases so use $(*)$

So, $I = \lim_{x\to-1+0}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{x^{m+k}}{m(m+k)} = \lim_{x\to-1+0}\frac{(\ln(1-x))^2}{2} = \frac{(\ln 2)^2}{2}$

$\therefore \frac{(\ln 2)^2}{2}$