POW 2009-11. Circles and Lines

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Does there exist a set of circles on the plane such that every line intersects at least one but at most 100 of them?

First, see the following figure. Figure.1 gives an bijective homeomorphism between a sphere and a plane.

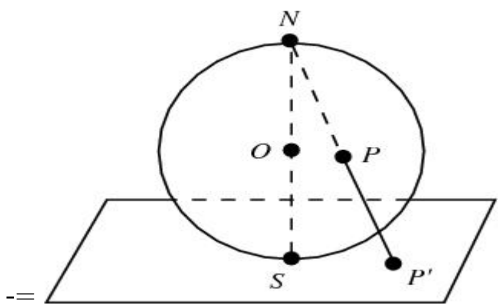

Figure 1: Streographic projection

The point N on the sphere is corresponded to the infinity of the plane. Then, we know the followings:
Fact1: A line on the plane is projected to a circle passing through N on the sphere.
Fact2: A circle on the plane is projected to a circle which does not pass through N on the sphere.

From now on, we are on the sphere where the plane is projected. So, if I say a line, then it indicates a circle passing through N on the sphere, and if I say a circle, then it is a circle which does not meet N on the sphere. Let E be a given set of circles such that every line intersects at least one but at most 100 of them. If E is finite, then d := min{dist(e, N) : e ∈ E} exists. Then, a line which is represented as a circle passing through N with radius d/2 on the sphere does noe meet any circle. This is a contradiction, so E is infinite. In particular, one has $\forall \epsilon > 0, \exists e \in E$ such that dist(e, N) $< \epsilon$. Then, there exists a sequene $\{e_n\}_{n \in \mathbb{N}}$ such that $\text{dist}(e_n, N)$ converges to 0. Now, let $l_1$be a line which is represented as a great circle on the sphere. Since l_1 intersects e_n for finitely many (in fact, at most 100) $n \in \mathbb{N}$, at least one of hemispheres divided by l_1 contains e_n for infinitly many $n \in \mathbb{N}$, call hemisphere H_1. Let l_2

be a great cirlce on the sphere that is orthogonal to l_1 at N. Since l_2 intersects e_n for finitely many $n \in \mathbb{N}$ and it divides H_1 into two equal region, one of the regions contains e_n for infinitly many $n \in \mathbb{N}$, call the region H_2. And, let l_3 be a great circle that divides H_2 into two equal regions. We can repeat this process infinitely. Since the angle between l_n and l_{n+1} goes to zero as n approaches to the infinity, l_n must converge to a great circle L. From this fact, one has that there exists a subsequence $\{c_n\}_{n\in\mathbb{N}}$ of $\{e_n\}_{n\in\mathbb{N}}$ such that both diameter of c_n and dist(c_n, N) are less than 1/n. Since at least one of hemispheres created by L contains infinitely many c_n's, then we can take a subsequence of $\{c_n\}_{n\in\mathbb{N}}$ which is on the one side of L. For convenience, just call that subsequence $\{c_n\}_{n\in\mathbb{N}}$.
Let γ be any smooth curve which passes through the centers of $c_1, c_2, \dots$ in order. Then, γ coverges to N, and it approaches to L, as it goes to N. Hence, γ must be tangent to L at N. Let D(r) be an open disc on the sphere centered at N with radius r. If r is small enough, the curvature of γ is almost constant in D(r). Let P_r be a circle which passes through N, is tangent to L at N, is located on the side where c_n's are on, has the curvature of γ in D(r). By the hypothesis, for any small $r > 0$, P_r intersects at most 100 c_n's. But, $\forall n \geq 2/r$, $c_n \in$D(r). Hence, we can take P_r to meet as many circles as we want with small enough r. In other words, for any $M \in \mathbb{N}$, $\exists r > 0$ such that $|\{n \in \mathbb{N} : P_r$ intersects $c_n\}| > M$. This shows that such set E of circles cannot exist, and the number of circles that a line intersects cannot be uniformly bounded.

Therefore, the answer to this questions is no.