1 problem

Let $a_0 = a$ and $a_{n+1} = a_n(a_n^2 - 3)$. Find all real values a such that the sequence $\{a_n\}$ converges.

2 solution

If $|a_n| > 2$, $|a_{n+1}| = |a_n||a_n^2 - 3| > |a_n|$. Otherwise $|a_n| \leq 2$ then $|a_{n+1}| = |a_n||a_n^2 - 3| \leq 2 \cdot 1 = 2$.

Therefore, $\{a_n\}$ never converges for any $|a| > 2$. Ignore them and assume $|a| \leq 2$ from now. (then $|a_n| \leq 2$ is also true.)

Substituting $a_n = 2\cos\theta_n$ ($\theta_n \in R$), We get

$$a_{n+1} = a_n(a_n^2 - 3) = 2\cos\theta_n(4\cos^2\theta_n - 3) = 2\cos 3\theta_n$$

(using $\cos^3\theta = (3\cos\theta + \cos 3\theta)/4$). Therefore, Letting $\theta = \arccos a/2$, we can deduce

$$a_n = 2\cos 3^n\theta.$$

Now, let us consider the exact limit values of $\{a_n\}$. We let $A = \lim a_n$, and from $\lim a_n = \lim a_{n+1}$ we can get $A = A(A^2 - 3)$, so $A = 0, 2,$ or -2. The corresponding angle values are $k\pi/2$ where $k \in Z$ (solutions of $\cos\theta = 0, 2, -2$.)

Define a sequence $\{a_n\}$ is **trivial** if there is a number t s.t.

$$a_t = a_{t+1} = a_{t+2} = \cdots .$$

We can prove $\{a_n\}$ is trivial iff there is a number t s.t. $a_t = 0, 2, -2$, from the equation $a_{t+1} = a_t$.

Firstly we find all values a s.t. $\{a_n\}$ converges trivially. From $a_t = 0, 2, -2$ and $a_t = 2\cos 3^t\theta$ we get

$$3^t\theta = k\pi/2$$

for an $k \in Z$. Thus

$$\theta = \frac{k\pi}{2 \cdot 3^t},$$

and from $a = 2\cos\theta$,

$$a = 2\cos\frac{k\pi}{2\cdot 3^t}.$$

that is all values for trivial convergence.

From now, we will prove that there is no **nontrivially** convergent sequence, that is, there is no θ such that $\{3^n\theta \mod 2\pi\}$ is nontrivially convergent to one of $k\pi/2$.

To make problem simply, first we try to find all θ to make $\{3^n\theta \mod \pi/2\}$ nontrivially convergent, or simply, to make

$$b_n = 3^n\theta/(\pi/2) \mod 1$$

nontrivially convergent.

Let $X = 2\theta/\pi$, and put

$$X = x + \sum_{i=1}^{\infty}\frac{x_i}{3^i}, x \in Z, x_i \in \{0,1,2\}.$$

(or simply, $X = x + 0.x_1x_2x_3\cdots_{(3)}$ in base 3.) Then the sequence is rewritten as

$$\begin{aligned} b_n &= 3^n\theta/(\pi/2) \mod 1 \\ &= 3^nX \mod 1 \\ &= 3^nx + \sum_{i=1}^{\infty} x_i\frac{3^n}{3^i} \mod 1 \\ &= \sum_{j=1}^{\infty}\frac{x_{n+j}}{3^j}. \end{aligned}$$

To make b_n convergent, there should be a number k s.t.

$$\forall i \geq 0 : x_{k+i} = m, m \in \{0,1,2\}.$$

Note that $x_{k+i} = 0$ makes the sequence trivial; $x_{k+i} = 2$ case is equivalent to $x_{k-1} = 1, x_{k+i} = 0$ because $0.222\cdots_{(3)} = 1$. Thus $m = 1$.

Then, $\sum_i 1/3^i = 1/2$ makes the equation into

$$b_{k-1} = \frac{1}{2} = 3^k\theta/(\pi/2)$$

$$\theta = \frac{\pi}{4 \cdot 3^k}$$

However, this values does not make $\{a_n\}$ convergent at all:

$$\begin{aligned} a_k &= 2\cos 3^k\theta \\ &= 2\cos 3^k \frac{\pi}{4 \cdot 3^k} \\ &= 2\cos\frac{\pi}{4} \\ &= 1 \end{aligned}$$

and

$$a_{k+1} = -1, a_{k+2} = 1, a_{k+3} = -1, \cdots ,$$

which is not a convergent sequence.

As shown above, we proved that there is no nontrivially convergent sequence.

Therefore, there are only trivially convergent sequences and corresponding a values are:

$$a = 2\cos\frac{k\pi}{2 \cdot 3^t}, k \in Z, t \in N$$