2009-6 Sum of integers of the fourth power

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Let x be an arbitrary positive integer. Then there exists $q \in \mathbb{Z}$ such that $x = 6q + \mathcal{R}$ where $\mathcal{R} \in \{0,1,2,3,4,5\}$. By Lagrange's Theorem, q can be expressed as the sum of four squares of integers. That is, $q = \mathcal{N}_1^{\,2} + \mathcal{N}_2^{\,2} + \mathcal{N}_3^{\,2} + \mathcal{N}_4^{\,2}$ and each $\mathcal{N}_i$ is an integer. To prove that x can be written as a sum of at most 53 biquadrates (which means fourth power of an integer), it is enough to show that every integer of the form $6\mathcal{N}^2$ can be written as a sum of 12 biquadrates. Again, by Lagrange's Theorem, $\mathcal{N}$ can be written as $\mathcal{N} = n_1^{\,2} + n_2^{\,2} + n_3^{\,2} + n_4^{\,2}$. Note that

$$6\mathcal{N}^2 = 6\left(\sum_{1\le i\le 4} n_i^{\,2}\right)^2 = 6\sum_{1\le i\le 4} n_i^{\,4} + 12\sum_{1\le i<j\le 4} n_i^{\,2}n_j^{\,2}$$
$$= \sum_{1\le i<j\le 4} (n_i + n_j)^4 + \sum_{1\le i<j\le 4} (n_i - n_j)^4$$

Since the number of 2-combination from the set of 4 elements is 6, the last representation implies that $6\mathcal{N}^2$ is written as the sum of 12 biquadrates. Because $\mathcal{R}$ can be written as a sum of at most five 1^4, each positive integer can be written as a sum of at most 53 biquadrates.