POW Problem 1 : Solution

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Consider matrix $A = [\frac{a_i^{j-1}}{(j-1)!}]$, that is,

$$A = \begin{pmatrix} 1 & a_1 & \frac{a_1^2}{2!} & \cdots & \frac{a_1^{n-1}}{(n-1)!} \\ 1 & a_2 & \frac{a_2^2}{2!} & \cdots & \frac{a_2^{n-1}}{(n-1)!} \\ 1 & a_3 & \frac{a_3^2}{2!} & \cdots & \frac{a_3^{n-1}}{(n-1)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \frac{a_n^2}{2!} & \cdots & \frac{a_n^{n-1}}{(n-1)!} \end{pmatrix}$$

It is very similar to Vandermonte's matrix $\Delta(a_1, \cdots, a_n) = [a_i^{j-1}]$. Note that

$$\det(\Delta(a_1, \cdots, a_n)) = \prod_{1 \le i < j \le n} (a_j - a_i).$$

Consider $\det(\Delta(a_1, \cdots, a_n))$ as a polynomial of $a_1, \ldots, a_n$. If we put the same value in a_i and a_j, then $\det(\Delta(a_1, \cdots, a_n)) = 0$ because the matrix have two same rows. So, $\det(\Delta(a_1, \cdots, a_n))$ is divisible by $a_j - a_i$. So, $\det(\Delta(a_1, \cdots, a_n))$ is divisible by $\prod_{1 \le i < j \le n}(a_j - a_i)$. And, both side have the same degree. Hence

$$\det(\Delta(a_1, \cdots, a_n)) = C \prod_{1 \le i < j \le n} (a_j - a_i).$$

for some constant C. By comparing the coefficents, we have $C = 1$. Similarly,

$$\det(A) = \frac{\prod_{1 \le i < j \le n}(a_j - a_i)}{\prod_{k=1}^{n-1} k!} = \prod_{1 \le i < j \le n} \frac{a_j - a_i}{j - i}.$$

Let $A_1 = A, A_2 = A$. From A_2, add the half of the second column to the third column and denote that matrix by A_3. Then, the entries of the third column are $\frac{a_1(a_1+1)}{2}, \ldots, \frac{a_n(a_n+1)}{2} \in \mathbb{Z}$. So, every entry of first three columns of A_3 is integer, and $\det(A_3) = \det(A_2) = \det(A)$. Inductively, define A_{i+1} from A_i by adding 'some' linear combination of first i columns of A_i to $(i+1)$-th column. The entries of the i-th column of A_i must be

$\frac{a_1(a_1+1)\ldots(a_1+i-2)}{(i-1)!}, \ldots, \frac{a_n(a_n+1)\ldots(a_n+i-2)}{(i-1)!} \in \mathbb{Z}$. Clearly $\det(A_i) = \det(A)$. So, $\det(A) = \det(A_n)$ where

$$A_n = \begin{pmatrix} 1 & a_1 & \frac{a_1(a_1+1)}{2!} & \cdots & \frac{a_1(a_1+1)\ldots(a_1+n-2)}{(n-1)!} \\ 1 & a_2 & \frac{a_2(a_2+1)}{2!} & \cdots & \frac{a_2(a_2+1)\ldots(a_2+n-2)}{(n-1)!} \\ 1 & a_3 & \frac{a_3(a_3+1)}{2!} & \cdots & \frac{a_3(a_3+1)\ldots(a_3+n-2)}{(n-1)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \frac{a_n(a_n+1)}{2!} & \cdots & \frac{a_n(a_n+1)\ldots(a_n+n-2)}{(n-1)!} \end{pmatrix}.$$

Note that A_n has only integer entries. So,

$$\det(A) = \prod_{1\leq i<j\leq n} \frac{a_j - a_i}{j - i}$$

is integer.