Problem of the week 2009-2: Sequence of Log

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Let us show first that $a_n \geq p_n$ for all $n \in \mathbb{N}$ where p_n is the n th prime number. ($p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\cdots$) Assume that there are at most $n-1$ distinct prime factors q_i divide $a_1 a_2 \cdots a_n$. Then we can write each a_i as $a_i = {q_1}^{\mathrm{r}_{i1}} {q_2}^{\mathrm{r}_{i2}} \cdots {q_{n-1}}^{\mathrm{r}_{i(n-1)}}$ for all $1 \leq i \leq n$. In other words, $\log a_i = \mathrm{r}_{i1} \log q_1 + \mathrm{r}_{i2} \log q_2 + \cdots + \mathrm{r}_{i(n-1)} \log q_{n-1}$. Notice that $\log q_i\,\mathrm{s}$ are linearly independent in the field $\mathbb{Q}$ because if $c_i \in \mathbb{Q}$, $\sum_{i=1}^{n} c_i \log q_i = 0$ means $\prod_{i=1}^{n} {q_i}^{c_i} = 1$ which again implies that $c_i = 0$ for all c_i. Now consider the equation $\sum_{i=1}^{n} \mathrm{b}_i \log a_i = 0$ about the b_i. This is consistent to the homogeneous equation

$$\begin{pmatrix} \mathrm{r}_{11} & \cdots & \mathrm{r}_{n1} \\ \vdots & \ddots & \vdots \\ \mathrm{r}_{1(n-1)} & \cdots & \mathrm{r}_{n(n-1)} \end{pmatrix} \begin{pmatrix} \mathrm{b}_1 \\ \mathrm{b}_2 \\ \vdots \\ \mathrm{b}_n \end{pmatrix} = 0$$

The rank of left matrix is, however, lower than the number of the unknowns, so there is a nontrivial solution $(\mathrm{b}_1, \mathrm{b}_2, \cdots, \mathrm{b}_n)$. This contradicts that $\log a_i\,\mathrm{s}$ are linearly independent over the rational field $\mathbb{Q}$. Therefore, there is at least n distinct prime factors divide $a_1 a_2 \cdots a_n$ and since $\{a_n\}$ is a strictly increasing sequence, we can assure that $a_n \geq p_n$.

Now by Rosser's Theorem, $p_n > n \log n$ for $n > 1$ (Rosser 1938). Thus,

$$\lim_{n\to\infty} \frac{a_n}{n} \geq \lim_{n\to\infty} \frac{p_n}{n} > \lim_{n\to\infty} \log n = \infty$$

which completes the proof.