Intersecting family.

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(*) Let $\mathcal{F}$ = collection of subsets (of size r) of a finite set E such that $X \cap Y \neq \phi$ for all $X, Y \in \mathcal{F}$.
Prove that $\exists\, S \subset E$ st $|S| \leq (2r-1) \cdot \binom{2r-3}{r-1}$ and
$X \cap Y \cap S \neq \phi$ for all $X, Y \in \mathcal{F}$.

proof) Let $S \subset E$ st $X \cap Y \cap S \neq \phi$ for all $X, Y \in \mathcal{F}$. ... $(*)$
And, let $\mathcal{F}_S := \{X \cap S : X \in \mathcal{F}\}$. Since E is finite, S is also finite, so let $|S| = m$.
Let $\{x_1, x_2, \cdots, x_m\}$ be any ordering of S.
Then, there is at most one i $(1 \leq i \leq m)$ such that both $\{x_1, \cdots, x_i\}$ and $\{x_i, \cdots, x_m\}$ contain $A, A' \in \mathcal{F}_S$, respectively.
To show this, suppose for $j \neq i$, $\{x_1, \cdots, x_j\}$ and $\{x_j, \cdots, x_m\}$ contain $B, B' \in \mathcal{F}_S$, respectively. Without loss of generality, we may assume $i < j$. Then, we have $A \cap B' = \phi$. But, this is a contradiction, since $A = X \cap S$ and $B' = Y \cap S$ for some $X, Y \in \mathcal{F}$ and then, $A \cap B' = X \cap Y \cap S \neq \phi$ by hypothesis.
Therefore, such j cannot exist. Now, we proved that for any ordering $\{x_1, \cdots, x_m\}$ of S, we can find at most one point x_i such that both $\{x_1, \cdots, x_i\}$ and $\{x_i, \cdots, x_m\}$ contain elements in $\mathcal{F}_S$.
Since there are $m!$ orderings, we have at most $m!$ number of such points. ... $(**)$.
Let S be a *minimal* one satisfying $(*)$. Then, for given $x \in S$, $\exists\, A, B \in \mathcal{F}_S$ such that $A \cap B = \{x\}$. It means, there is an ordering $\{x_1, \cdots, x_m\}$ of S such that $x = x_i$ for some i, and $\{x_1, \cdots, x_i\} \supset A$ & $\{x_i, \cdots, x_m\} \supset B$. Thus, we can see that every element of S is counted many times. Set $|A| = a$ and $|B| = b$ here. I will make such ordering in this way;
First, put the elements of $A \setminus \{x\}$, and put x, and then put the elements of $B \setminus \{x\}$. At last, put the remaining elements of S arbitrarily.
How many ways are there to do this? $\exists\, (a-1)!$ ways to arrange the elements of $A \setminus \{x\}$, and $\exists\, (b-1)!$ ways to arrange the elements of $B \setminus \{x\}$. And, $|S \setminus (A \cup B)| = m - a - b + 1$.

So, $\exists\ (m-a-b+1)!$ ways to arrange the remaining elements. And, A, B can be chosen $\binom{m}{a+b-1}$ different ways. Therefore, total $(a-1)!\,(b-1)!\,(m-a-b+1)!\binom{m}{a+b-1}$ ways are possible. $(m-a-b+1)!\binom{m}{a+b-1} = \frac{m!}{(a+b-1)!}$ so simply, $(a-1)!\,(b-1)!\,\frac{m!}{(a+b-1)!}$ ways.

Note that $a, b \le r$. From this fact, we can see that

$$\frac{(a+b-1)!}{(a-1)!\,(b-1)!} = a\cdot\frac{(a+(b-1))!}{a!\,(b-1)!} = a\cdot\binom{a+b-1}{b-1} \le r\cdot\binom{2r-1}{r-1}.$$

Then, $m!\,\frac{(a-1)!\,(b-1)!}{(a+b-1)!} \ge \frac{m!}{r\cdot\binom{2r-1}{r-1}}$. Furthermore, we can change the role of A and B. Thus, any element in S is counted at least $\frac{2m!}{r\binom{2r-1}{r-1}}$ times among the $m!$ possibilities. (see (**).) Therefore, $|S| \le m! \Big/ \left(\frac{2m!}{r\cdot\binom{2r-1}{r-1}}\right)$

$$= \frac{1}{2}\cdot r\cdot\binom{2r-1}{r-1} = \frac{r}{2}\cdot\frac{(2r-1)\cdot(2r-2)}{r\cdot(r-1)}\cdot\binom{2r-3}{r-1} = (2r-1)\binom{2r-3}{r-1}.$$

In conclusion, we can find such S with $|S| \le (2r-1)\cdot\binom{2r-3}{r-1}$.

It's all done. $\blacksquare$