POW #12 : Solution

Yoon Haewon, KAIST 04

2008.12.1

$$\int_0^1 \min(x, y) f(y) dy = \lambda f(x)$$

The following is a obvious fact.

Claim 1 *If $f = 0$ on $[0,\ 1]$, then $f = 0$ everywhere.*

Suppose $\lambda \neq 0$.

Claim 2 *f is differentiable on $(0,\ 1)$ and*

$$f'(x) = \frac{1}{\lambda} \int_x^1 f(y) dy.$$

proof. Remind that if f is continuous on the closed interval then f is bounded on that interval.

$$\begin{aligned}
\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0^+} \frac{1}{h\lambda} \int_0^1 (\min(x+h, y) - \min(x, y)) f(y) dy \\
&= \lim_{h \to 0^+} \frac{1}{h\lambda} \left(\int_x^{x+h} (y - x) f(y) dy + \int_{x+h}^1 h f(y) dy \right) \\
&= \lim_{h \to 0^+} \frac{1}{h\lambda} \int_{x+h}^1 h f(y) dy \\
&= \frac{1}{\lambda} \int_x^1 f(y) dy.
\end{aligned}$$

(Second equality follows from the fact that $0 \leq |(y-x)f(y)| \leq \max(|M|, |m|)h$ on $[x,\ x+h]$ where M, m are the maximum and the minimum of f on $[x,\ x+h]$.)

Similarly,

$$\begin{aligned}
\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0^-}\frac{1}{h\lambda}\int_0^1(\min(x+h,y)-\min(x,y))f(y)dy \\
&= \lim_{h\to 0^-}\frac{1}{h\lambda}\left(\int_{x+h}^{x}(x+h-y)f(y)dy+\int_x^1 hf(y)dy\right) \\
&= \lim_{h\to 0^-}\frac{1}{h\lambda}\int_{x+h}^{1}hf(y)dy \\
&= \frac{1}{\lambda}\int_x^1 f(y)dy. \ \spadesuit
\end{aligned}$$

Claim 3 *f' is differentiable on* $(0,\ 1)$ *and*

$$f''(x) = -\frac{1}{\lambda}f(x)$$

Proof. From the fundamental Theorem of Calculus, it's almost obvious. $\spadesuit$

Now we have to solve the second order ordinary differential equation

$$\lambda f''(x) + f(x) = 0.$$

If $\lambda = \frac{1}{\mu^2} > 0$ and $\mu > 0$, the general Solution of this equation is

$$f(x) = c_1\cos(\mu x) + c_2\sin(\mu x)$$

Since f is continuous, it must satisfy $f(0) = 0$ and $f(1) = \mu^2\int_0^1 yf(y)dy$.

$$f(0) = 0 \qquad \Longrightarrow \qquad c_1 = 0$$

$$\begin{aligned}
f(1) = \mu^2\int_0^1 yf(y)dy \quad &\Longrightarrow \quad c_2\sin(\mu) = c_2\mu^2\int_0^1 y\sin(\mu y)dy \\
&\Longrightarrow \quad \mu\cos(\mu)c_2 = 0
\end{aligned}$$

If $c_2 = 0$, $f = 0$. Since $\mu \neq 0$, $\cos\mu = 0$. Hence eigenvalue λ's are $\frac{1}{\left(n\pi+\frac{\pi}{2}\right)^2}$ $(n \in \mathbb{N}\cup\{0\})$. And corresponding eigenvector

$$f(x) = C\sin\left(\left(n\pi+\frac{\pi}{2}\right)x\right)$$

for any constant C.

If $\lambda = -\frac{1}{\mu^2} < 0$ and $\mu > 0$, the general Solution of this equation is

$$f(x) = d_1e^{\mu x} + d_2e^{-\mu x}$$

Since f is continuous, it must satisfy $f(0) = 0$ and $f(1) = \mu^2 \int_0^1 yf(y)dy$.

$$f(0) = 0 \qquad \Longrightarrow \qquad d_1 + d_2 = 0$$

$$\begin{aligned} f(1) = -\mu^2 \int_0^1 yf(y)dy \quad &\Longrightarrow \quad d_1\left(e^{\mu} - e^{-\mu}\right) = -d_1\mu^2 \int_0^1 y\left(e^{\mu y} - e^{-\mu y}\right)dy \\ &\Longrightarrow \quad \mu\left(e^{\mu} + e^{-\mu}\right)d_1 = 0 \end{aligned} \tag{1}$$

Since $e^{\mu} + e^{-\mu} > 0$ and $\mu > 0$, $d_1 = 0$. Hence $f = 0$.

Finally, consider the case when $\lambda = 0$

Since f is continuous on $[0, 1]$, there exist second order antiderivative function F on $(0, 1)$. $(F'' = f)$ [I]

$$\lambda F''(x) = \int_0^x yF''(y)dy + x\int_x^1 F''(y)dy$$

$$\lambda F^{''}(x) = \left[yF'(y)\right]_0^x - \int_0^x F'(y)dy + x\int_x^1 F^{''}(y)dy$$

$$\lambda F^{''}(x) = xF'(x) - (F(x) - F(0)) + x(F'(1) - F'(x))$$

$$\lambda F^{''}(x) = F(0) - F(x) + F'(1)x$$

$$\lambda F^{''}(x) + F(x) - F'(1)x - F(0) = 0^{\text{II}}$$

If $\lambda = 0$, $F(x) = F'(1)x - F(0)$. Hence $f(x) = F''(x) = 0$.

Now we have to determine the value of f on $(-\infty, 0) \cup (1, \infty)$
If $x \in (-\infty, 0)$, then

$$\lambda f(x) = x\int_0^1 f(y)dy$$

[I] It is also a corollary of the fundamental theorem of calculus.

[II] If we solve this differential equation, then we get the same solutions when $\lambda \neq 0$. That means it can be another way to solve this problem.

$$\begin{aligned} f(x) &= \left(n\pi + \frac{\pi}{2}\right)^2 \left(\int_0^1 C \sin\left(\left(n\pi + \frac{\pi}{2}\right) y\right) dy\right) x \\ &= C\left(n\pi + \frac{\pi}{2}\right) x \qquad (2) \end{aligned}$$

If $x \in (0, \infty)$, then

$$\lambda f(x) = \int_0^1 y f(y) dy = f(1) = C \sin\left(n\pi + \frac{\pi}{2}\right) = C(-1)^n$$

Answer. Eigenvalues λ's are $\frac{1}{\left(n\pi + \frac{\pi}{2}\right)^2}$ $(n \in \mathbb{N} \cup \{0\})$. And corresponding eigenvector

$$f(x) = \begin{cases} C\left(n\pi + \frac{\pi}{2}\right) x & \text{if } x \in (-\infty,\, 0). \\ C \sin\left(\left(n\pi + \frac{\pi}{2}\right) x\right) & \text{if } x \in [0,\, 1]. \\ C(-1)^n & \text{if } x \in (1,\, \infty). \end{cases}$$

for all constant C.