양해훈 POW#2008-9

Lemma. Function $g: Z \Rightarrow N, g(x) = \begin{cases} 2x & x > 0 \\ -2x+1 & x \leqq 0 \end{cases}$ is injective function.

Proof of Lemma

Let $g(x) = g(y) = a$ for some integers x, y, a. a can be even or odd number.

i) a is even: x and y must be positive. $g(x) = g(y)$ yields $2x = 2y$, thus $x = y$.

ii) a is odd: x and y must not be positive. From $g(x) = g(y)$, $-2x+1 = -2y+1$, and $x = y$.

Therefore, g is injective.

Let $h: R^2 \Rightarrow N, h(x,y)$ be defined as the least positive number N that satisfies $\lfloor Nx \rfloor \neq \lfloor Ny \rfloor$ or 1 if $x = y$, and define f as

$$f(x,y,z) = \begin{cases} 1 & x = y = z \\ 5 & x = y \neq z \\ 7 & x \neq y = z \\ 2^{h(x,y)} \times 3^{g(\lfloor xh(x,y) \rfloor)} & Otherwise \end{cases}$$

then f is a function from R^3 to N. I will prove that $f(x,y,z) = f(y,z,w)$ then $x = y = z = w$. There are only four kinds of values $f(x,y,z)$ can have.

i) $f(x,y,z) = f(y,z,w) = 1$: Obviously $x = y = z = w$.

ii) $f(x,y,z) = f(y,z,w) = 5$: From $f(x,y,z) = 5$, $y \neq z$ and from $f(y,z,w) = 5$, $y = z$. Contradiction.

iii) $f(x,y,z) = f(y,z,w) = 7$: From $f(x,y,z) = 7$, $y = z$ and from $f(y,z,w) = 7$, $y \neq z$. Contradiction.

iv) $f(x,y,z) = f(y,z,w) = 2^{\alpha} \times 3^{\beta}$ for some positive numbers α, β:

Note that $x \neq y$ and $y \neq z$, so always $\lfloor xh(x,y) \rfloor \neq \lfloor yh(x,y) \rfloor$ this case.

Because $2^{h(x,y)} \times 3^{g(\lfloor xh(x,y) \rfloor)} = 2^{h(y,z)} \times 3^{g(\lfloor yh(y,z) \rfloor)}$, $h(x,y) = h(y,z)$ and $g(\lfloor xh(x,y) \rfloor) = g(\lfloor yh(y,z) \rfloor)$ and by Lemma, $\lfloor xh(x,y) \rfloor = \lfloor yh(y,z) \rfloor$. Therefore, $\lfloor xh(x,y) \rfloor = \lfloor yh(x,y) \rfloor$.

We already showed $\lfloor xh(x,y) \rfloor \neq \lfloor yh(x,y) \rfloor$, and it is contradiction.

By i) ~ iv), f is the function that satisfies the condition of the problem.