구건방운

28112400

@naver.com

(*) Let A be an $n\times n$ matrix

$\lambda_1, \lambda_2, \cdots, \lambda_p$ are the eigenvalues of A with multiplicity $r_1, \cdots, r_p$, respectively. Then

$\det(A) = \lambda_1^{r_1} \lambda_2^{r_2} \cdots \lambda_p^{r_p}$

$tr(A) = r_1\lambda_1 + r_2\lambda_2 + \cdots + r_p\lambda_p$

(proof) Consider $\det(A - xI)$

$$= \begin{vmatrix} a_{11}-x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & & a_{nn}-x \end{vmatrix}$$

The only way an x^{n-1} can be obtained is from a product of $n-1$ elements of the diagonal elements, multiplied by the scalar from the remaining diagonal element. Thus, the coefficient of x^{n-1} is $(-1)^{n-1}\sum_{i=1}^{n} a_{ii} = (-1)^{n-1} tr(A)$

Let $f(x) = \det(A - xI) = (x-\lambda_1)^{r_1} \cdots (x-\lambda_p)^{r_p}(-1)^n$

Then $\det(A) = f(0) = (-1)^{2n}\lambda_1^{r_1}\cdots\lambda_p^{r_p} = \lambda_1^{r_1}\cdots\lambda_p^{r_p}$

$(r_1 + r_2 + \cdots + r_p = n)$

$tr(A) = (-1)(-r_1\lambda_1 - r_2\lambda_2 - \cdots - r_n\lambda_n)$

$= r_1\lambda_1 + \cdots + r_n\lambda_n$ ▨

Now I'll show the statement of the problem.

$$1 = \frac{1}{n}(\underbrace{1+1+\cdots+1}_{n\text{-times}}) \qquad (\because \text{all } A_{ii} = 1 \text{ or } 0)$$

$$\geq \frac{1}{n}\operatorname{tr}(A)$$

$$= \frac{1}{n}(r_1\lambda_1 + \cdots + r_p\lambda_p)$$

$$\geq (\lambda_1^{r_1}\lambda_2^{r_2}\cdots\lambda_p^{r_p})^{\frac{1}{n}} \qquad (\text{AM} \geq \text{GM})$$

$$= (\det A)^{\frac{1}{n}} \qquad (\det A > 0 \text{ and it is an integer})$$

$$\geq 1.$$

Thus, all equals hold, so

$\lambda_1 = \cdots = \lambda_p = 1$. We are done. $\blacksquare$