권상훈)

28112400@naver.com

(proof)

I will use the mathematical induction to prove this inequality. This inequality is equivalent to

$$f(a_1, a_2, \cdots, a_n) = (n-1)\sum_{i=1}^{n} a_i^n + n\prod_{i=1}^{n} a_i - \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} a_i^{n-1}\right) \geq 0$$

Since f is symmetric and homogeneous. we may assume that $a_1 \geq \cdots \geq a_n \geq a_{n+1}$ and $a_1 + \cdots + a_n = 1$. Now, for $n=1$, $a_1 \geq a_1$ (trivial)

Suppose the inequality is true for n a_i's, and let us show the case for $n+1$ numbers.

We need to prove

$$n\sum_{i=1}^{n+1} a_i^{n+1} + (n+1)\prod_{i=1}^{n+1} a_i - \left(\sum_{i=1}^{n+1} a_i\right)\left(\sum_{i=1}^{n+1} a_i^n\right) \geq 0$$

which is equivalent to

$$n\sum_{i=1}^{n} a_i^{n+1} + n a_{n+1}^{n+1} + n a_{n+1}\prod_{i=1}^{n} a_i + a_{n+1}\prod_{i=1}^{n} a_i - (1 + a_{n+1})\left(a_{n+1}^n + \sum_{i=1}^{n} a_i^n\right) \geq 0 \quad \cdots (*)$$

(Next Page →)

By the inductive hypothesis, we obtain

$$(n-1)\sum_{i=1}^{n} a_i^{n} + n\prod_{i=1}^{n} a_i \geq \sum_{i=1}^{n} a_i^{n-1}$$

$$\Rightarrow \quad n\prod_{i=1}^{n} a_i \geq \sum_{i=1}^{n} a_i^{n-1} - (n-1)\sum_{i=1}^{n} a_i^{n}$$

$$\Rightarrow \quad n\prod_{i=1}^{n+1} a_i \geq a_{n+1}\sum_{i=1}^{n} a_i^{n-1} - (n-1)a_{n+1}\sum_{i=1}^{n} a_i^{n} \quad (\because a_{n+1}\geq 0)$$

Put this inequality to (*), we have

$$n\sum_{i=1}^{n} a_i^{n+1} + n a_{n+1}^{n+1} + a_{n+1}\sum_{i=1}^{n} a_i^{n-1} - (n-1)a_{n+1}\sum_{i=1}^{n} a_i^{n} + \prod_{i=1}^{n+1} a_i$$
$$- \left(\underline{1+a_{n+1}}\right)\left(\underline{a_{n+1}^{n}} + \underline{\sum_{i=1}^{n} a_i^{n}}\right) \geq 0,$$

or equivalently,

$$\left\{\left(n\sum_{i=1}^{n} a_i^{n+1} - \underline{\sum_{i=1}^{n} a_i^{n}}\right) - a_{n+1}\left(n\underline{\sum_{i=1}^{n} a_i^{n}} - \sum_{i=1}^{n} a_i^{n-1}\right)\right\}$$
$$+ a_{n+1}\left\{\underline{(n-1)a_{n+1}^{n}} + \prod_{i=1}^{n} a_i - \underline{a_{n+1}^{n-1}}\right\} \geq 0$$

※ Claim1: $\underline{\left(n\sum_{i=1}^{n} a_i^{n+1} - \sum_{i=1}^{n} a_i^{n}\right) - a_{n+1}\left(n\sum_{i=1}^{n} a_i^{n} - \sum_{i=1}^{n} a_i^{n-1}\right) \geq 0}$ ①

($\because$ Since $a_{n+1} \leq \frac{1}{n}$ and

$$n\sum_{i=1}^{n} a_i^{n} - \sum_{i=1}^{n} a_i^{n-1} = n^2\left(\frac{1}{n}\sum_{i=1}^{n} a_i a_i^{n-1} - \frac{1}{n^2}\sum_{i=1}^{n} a_i \sum_{i=1}^{n} a_i^{n-1}\right) \geq 0$$

(by Chebychev)

it suffices to show

$$n\sum_{i=1}^{n} a_i^{n+1} - \sum_{i=1}^{n} a_i^{n} \geq \frac{1}{n}\left(n\sum_{i=1}^{n} a_i^{n} - \sum_{i=1}^{n} a_i^{n-1}\right)$$

(Next →)

Since $na_i^{n+1} + \frac{1}{n} a_i^{n-1} \geqq 2a_i^n$ for $\forall i$. (AM-GM)

$$\sum_{i=1}^{n} \left(n a_i^{n+1} + \frac{1}{n} a_i^{n-1} \right) \geqq 2 \sum_{i=1}^{n} a_i^n$$

$$\Leftrightarrow \quad n \sum_{i=1}^{n} a_i^{n+1} - \sum_{i=1}^{n} a_i^n \geqq \frac{1}{n} \left(n \sum_{i=1}^{n} a_i^n - \sum_{i=1}^{n} a_i^{n-1} \right) _{///})$$

※ Claim 2 : $\underline{a_{n+1} \left((n-1) a_{n+1}^n + \prod_{i=1}^{n} a_i - a_{n+1}^{n-1} \right) \geqq 0}$ ②

(∵ $(n-1) a_{n+1}^n + \prod_{i=1}^{n} a_i - a_{n+1}^{n-1}$

$$= (n-1) a_{n+1}^n + \prod_{i=1}^{n} (a_i - a_{n+1} + a_{n+1}) - a_{n+1}^{n-1}$$

$$\geqq (n-1) a_{n+1}^n + a_{n+1}^n + a_{n+1}^{n-1} \sum_{i=1}^{n} (a_i - a_{n+1}) - a_{n+1}^{n-1}$$

$$= n a_{n+1}^n + a_{n+1}^{n-1} (1 - n a_{n+1}) - a_{n+1}^{n-1} = 0$$

($\prod_{i=1}^{n} (a_i - a_{n+1} + a_{n+1}) = a_{n+1}^n + a_{n+1}^{n-1} \sum_{i=1}^{n} (a_i - a_{n+1}) + \cdots$

we get this by expansioning directly) $_{///}$)

Hence, (*) is obvious from ① + ② by claim 1, 2. Thus the original inequality is proven.

(proof is over)